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Consistent Off-Shell String Amplitudes

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Abstract

Based on the covariant loop calculus developed by Di Vecchia et al. and by Petersen et al., we construct consistent, modular-invariant, off-shell closed string amplitudes. Our prescription resolves improper non-localities in the would-be off-shell amplitudes of the covariant loop calculus, thereby eliminating on-shell ambiguities in corners of punctured moduli space and deriving consistent renormalization constants in the infinite-tension limit. We also show that, contrary to expectations, the on-shell limit of the amplitudes in general string models depends on the fixing of the projective freedom inherent in the covariant loop calculus.

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1. Introduction

Most of our limited knowledge of string theory is contained in prescriptions for computing scattering amplitudes for various on-shell physical external states. The two main approaches to the construction of scattering amplitudes are the path-integral approach pioneered by Polyakov [1] and the operator approach [2]. Amongst the plethora of constructions, Di Vecchia et al. [3] (in an operator formalism) and Petersen et al. [4] (in a path-integral formalism) have developed an approach based on the notion of sewing together three-string vertices into tree or loop amplitudes for arbitrary numbers of external string states. This is the so-called covariant loop calculus. What is interesting about their approach is that it appears, at first sight, to offer an extension from on-shell to off-shell string amplitudes. The off-shell extension introduces a projective invariance associated with each external leg.

When the external states are put on-shell, the factors in the amplitude associated with the projective freedom formally reduce to unity, and the amplitudes become *formally* identical to those obtained in the standard Polyakov approach. The equivalence is straightforward at the one-loop level, and presumably holds at higher loops as well [5].

This equivalence is not necessarily a good thing, however. The problem is that Polyakov amplitudes are in general ill-defined [6,7,8,9]. In corners of punctured moduli space where loops are isolated on external legs, the amplitudes receive contributions which contain an on-shell propagator. For massless-vector external states, the numerator also vanishes (because of gauge invariance), and so these contributions to the amplitudes contain a '0/0', which is to say, they are ill-defined. This difficulty shows up in all bosonic string theories, and in all superstring or heterotic-string theories with at least one broken space-time supersymmetry. (Massive external states require a regularization and renormalization scheme, such as that discussed by Weinberg [10].)

Even off-shell, as we shall see, the formalisms of refs. [3,4] in general yield expressions containing improper non-localities; the on-shell limit, when taken carefully, reveals not only a failure to resolve the ambiguity in the Polyakov amplitudes, but an 'essential singularity' in the momentum invariants to boot. It comes as no surprise that the low-energy limits of these would-be off-shell string amplitudes are not off-shell amplitudes of a gauge field theory, but are pathological as well.

As we shall show, it is possible to cure this difficulty, and thereby resolve the ambiguity in on-shell Polyakov amplitudes in a fashion known to be consistent with gauge invariance and unitarity, by making special choices of the projective transformations associated with the external legs in the covariant loop calculus. This choice yields a set of off-shell amplitudes which are sensible, consistent, and (as it turns out) modular-invariant.

In an ordinary field theory different choices of gauge or field variables will yield different sets of renormalization constants. In the string case we are dealing with sets of amplitudes which are not derived from an underlying string field theory, so care must be taken to ensure that the various amplitudes yield a consistent set of renormalization constants; our choice does. This is, loosely speaking, equivalent to requiring all loop amplitudes to be defined using the same field variables. The algebraic structure of these amplitudes is similar enough to that of Polyakov amplitudes (after resolving the ambiguity in the latter) that one can readily adapt the calculation of renormalization constants [6,7,8,9] to the off-shell case. The off-shell amplitudes we will present can therefore be regarded as the amplitudes that would emerge from a consistent string field theory.

In this paper, we will demonstrate explicitly that our choice of the projective transformations in the covariant loop calculus amplitudes resolves the momentum/pole ambiguities for the two-, three-, and four-point one-loop amplitudes with massless external states. We also present an argument valid for arbitrary numbers of external legs. The choices of projective transformations fall into a simple pattern which allows for a generalization to all M-loop, N-massless vector amplitudes. A key underlying principle for our scheme is off-shell modular invariance which, though not required by unitarity as is the corresponding on-shell invariance, is nonetheless natural for a closed string.

Although alternative operator formalisms exist for constructing string loop amplitudes [11], it appears that without the significant modifications necessary to exhibit the projective invariance manifestly, they would not admit similar prescriptions. For string field theories, the requirement is more stringent: candidate theories must resolve the on-shell ambiguities, and resolve them in a fashion consistent with gauge invariance, or else they are simply wrong. (Although we will not discuss open-string amplitudes, similar ambiguities appear in that case, and procedures similar to ours may be used to resolve them. Open-string field theories must likewise resolve them.)

The present work grew out of the calculation of renormalization constants in the string-based approach to perturbative QCD [7,9]. It is possible, using a prescription due to Minahan [6], which violates momentum conservation in favor of preserving modular invariance, to resolve the ambiguities entirely within the framework of on-shell amplitudes; and one can demonstrate that this procedure gives the correct answer [9]. An off-shell formalism such as that presented in this paper is thus not necessary for practical calculations. But the Minahan procedure is nonetheless rather mysterious, and the off-shell constructions in this paper can explain why it works.

In the next section, we present the one-loop amplitudes obtained from the covariant loop calculus [3,4], emphasizing their projective and super-projective invariances. In section 3, we review the ambiguity in Polyakov amplitudes that arises in the regions of punctured moduli space where loops are isolated on external legs, while in section 4, we examine the related difficulties in the amplitudes of the covariant loop calculus. In section 5, we present a choice of projective transformations which resolve these difficulties. In section 6, we show how this choice works in the three-point amplitude, while the corresponding resolution in the four-point amplitude is discussed in section 7. In section 8, we present a general argument for the N-point amplitude. This choice yields a consistent set of renormalization constants, as we demonstrate in section 9 by calculating the two-point function. In section 10, we present a more general set of choices for the projective transformations that also resolve the ambiguities and non-localities. Appendices I and II contain summaries of the covariant loop calculus for bosonic and super- strings, respectively. In appendix III we give a brief review of the fermionic formulation of four-dimensional heterotic strings, while in appendix IV the reader will find our conventions for the closed string used in the body of the paper.

2. One-loop Amplitudes in the Covariant Loop Calculus

We will consider amplitudes in heterotic string theories using the fermionic formulation [12] (see Appendix III for a brief review). Taking the standard one-loop Polyakov amplitude, and adding the terms dictated by the covariant loop calculus (see Appendices I and II), the N-point massless vector amplitude in a four-dimensional heterotic string becomes

$$
\begin{aligned}
\mathcal{A}_N = {} & \frac{1}{2(16\pi^2)}\lambda^{N/2-2}(2g)^N\, T^{a_1}{}_{m_1}{}^{n_1}\cdots T^{a_N}{}_{m_N}{}^{n_N} \\
& \int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\times\int\left(\prod_{i=1}^{N}d\theta_{i1}\,d\theta_{i2}\,d\theta_{i3}\,d\theta_{i4}\right)\mathrm{Im}\,\tau\int\left(\prod_{i=1}^{N-1}d^2\nu_i\right)\sum_{\vec\alpha,\vec\beta}C^{\vec\alpha}_{\vec\beta}\,Z\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau) \\
& \prod_{i=1}^{N}\Bigg[\left(e^{-2\pi i\nu_i}V_i'(0)\right)^{-\lambda k_i^2/4\pi}\left(e^{i\pi\overline{\nu}_i}\overline{D}\overline{V}_i^F(0)\right)^{-\lambda k_i^2/2\pi} \\
& \qquad\qquad \times\exp\left(-i\sqrt{\lambda}\theta_{i4}\varepsilon_i\cdot k_i\Big(\frac{e^{-i\pi\overline{\nu}_i}}{\sqrt{-2\pi i}}\frac{\overline{D}^2\overline{V}_i^F(0)}{(\overline{D}\overline{V}_i^F(0))^2}+i\frac{\theta_{i3}}{2}\Big)\right)\Bigg] \\
& \times\prod_{i<j}\exp\Bigg[\lambda k_i\cdot k_j\,G_B(\nu_i-\nu_j)-\theta_{i3}\theta_{j3}\lambda k_i\cdot k_j\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad -\theta_{i1}\theta_{j2}\delta^{m_i}{}_{n_j}G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{n_j}\end{bmatrix}(\nu_i-\nu_j)-\theta_{i2}\theta_{j1}\delta^{m_j}{}_{n_i}\hat{G}_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{n_i}\end{bmatrix}(\nu_i-\nu_j) \\
& \qquad +i\sqrt{\lambda}(\theta_{i3}\theta_{j4}k_i\cdot\varepsilon_j+\theta_{i4}\theta_{j3}k_j\cdot\varepsilon_i)\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad -i\sqrt{\lambda}(\theta_{i3}\theta_{i4}k_j\cdot\varepsilon_i-\theta_{j3}\theta_{j4}k_i\cdot\varepsilon_j)\,\dot{G}_B(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad +\theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j)+\theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\,\ddot{G}_B(\overline{\nu}_i-\overline{\nu}_j)\Bigg]
\end{aligned}
\tag{2.1}
$$

The integrations over the Grassmann parameters θ_{iJ} automatically select the terms multi-linear in the polarization vectors. The parameters θ_{i3} also play the role of supercoördinates for the right-mover superstring (see Appendix II). The modular parameter, τ, is integrated over the fundamental modular region of the torus, while the Koba-Nielsen variables ν_i run over the torus specified by τ. The vectors $\vec{\alpha}$ and $\vec{\beta}$ specify the boundary conditions of the world sheet fermions on the torus with $\alpha_\dagger$ representing the boundary conditions of the world sheet fermions carrying space-time indices. Polchinski [13] and Sakai and Tanii [14] have calculated the over-all normalization. The inverse string tension is $\lambda = \pi\alpha'$, and our conventions for closed string Green functions on the torus are given in Appendix IV. (These conventions are designed to minimize the number of πs shuffled around in explicit calculations.) All calculations in the body of this paper are in Minkowski space with metric $(+\ -\ -\ -)$. Convergence is provided where necessary by rotation to Euclidean space.

For sectors containing Ramond zero-modes there are technical complications which alter the form of the amplitude (2.1). However, these sectors contribute only to parity-violating amplitudes, which are not relevant to any of our discussions.

The V_is embody the projective invariance inherent in the amplitude, and they play a central role in our constructions. The projective transformation functions for the bosonic left-movers are

$$V_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \qquad\qquad a_i d_i - b_i c_i = 1 \tag{2.2}$$

while for the superstring right-movers the superprojective transformations [15] are

$$\overline{V}_i(\overline{Z}) = (\overline{V}_i^B(\overline{Z}), \overline{V}_i^F(\overline{Z})) \tag{2.3}$$

where $\overline{Z} = (\bar{z}, \bar{\theta})$ is a point in superspace and

$$\left.\begin{aligned} \overline{V}_i^B(\overline{Z}) &= \frac{\bar{a}_i \bar{z} + \bar{b}_i}{\bar{c}_i \bar{z} + \bar{d}_i} + \bar{\theta}\frac{\bar{\gamma}_i \bar{z} + \bar{\delta}_i}{(\bar{c}_i \bar{z} + \bar{d}_i)^2} \\ \overline{V}_i^F(\overline{Z}) &= \frac{\bar{\gamma}_i \bar{z} + \bar{\delta}_i}{\bar{c}_i \bar{z} + \bar{d}_i} + \bar{\theta}\frac{1 + \frac{1}{2}\bar{\delta}_i \bar{\gamma}_i}{\bar{c}_i \bar{z} + \bar{d}_i} \end{aligned}\right\} \qquad \bar{a}_i \bar{d}_i - \bar{b}_i \bar{c}_i = 1 \tag{2.4}$$

The covariant loop calculus construction requires the projective transformations to satisfy

$$V_i(0) = e^{2\pi i \nu_i} \tag{2.5}$$

while the superprojective transformations are required to satisfy

$$\overline{V}_i(0,0) = (e^{-2\pi i \overline{\nu}_i}, -i\sqrt{-2\pi i}\, e^{-i\pi \overline{\nu}_i}\theta_{i3}) \tag{2.6}$$

As discussed in Appendix II the factor $\sqrt{-2\pi i}\, e^{-i\pi\overline{\nu}}$ associated with θ_{i3} arises from the Jacobian of the transformation from the $\bar{z}_i$ to the usual torus variables $\overline{\nu}_i = -\ln \bar{z}_i / 2\pi i$, while the factor of

i arises from differences in the conventions for the Grassmann variables. In addition, there is an overall complex conjugation for the right-movers of the closed string. These constraints leave us with two complex and one Grassmannian degrees of freedom. At this stage, it would appear that we can choose these at will without affecting the on-shell limit, $k_i^2 = 0$ and $k_i \cdot \varepsilon_i = 0$. As we shall see later, that is not correct.

For the expressions appearing in the amplitude (2.1), these various constraints give us

$$V_i'(0) = \left.\frac{\partial V_i(z)}{\partial z}\right|_{z=0} = d_i^{-2} \tag{2.7}$$

$$\begin{aligned}\overline{DV}_i^F(0) &= \left(\frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar z}\right)\overline{V}_i(\bar z,\bar\theta)\Big|_{\bar z=0,\bar\theta=0} \\ &= \bar d_i^{-1} - i\sqrt{-2\pi i}e^{-i\pi\overline{\nu}_i}\theta_{i3}\bar\gamma_i/2\end{aligned} \tag{2.8}$$

and

$$\frac{\overline{D}^2\overline{V}_i^F(0)}{(\overline{DV}_i^F(0))^2} = \bar d_i(\bar\gamma_i + i\sqrt{-2\pi i}e^{-i\pi\overline{\nu}_i}\bar c_i\theta_{i3}) \tag{2.9}$$

We will eventually wish to choose these constants so as to make the amplitude (2.1) well-defined. First, however, we must establish the need to do so; that is the topic of the next two sections.

3. Ambiguity in Polyakov Amplitudes

In this section, we review the ambiguity that arises in Polyakov amplitudes [6,9]. This ambiguity occurs for amplitudes with any number of external massless vectors, but is explained most easily in the three-vector amplitude. Putting the external momenta on shell in equation (2.1), the $\varepsilon_1 \cdot \varepsilon_2 \,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})$ term in the Polyakov three-point amplitude is

$$\begin{aligned}
&i\frac{4g^3}{16\pi^2}\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})\varepsilon_1\cdot\varepsilon_2\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu_1 d^2\nu_2\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta}C^{\vec\alpha}_{\vec\beta}Z\begin{bmatrix}\vec\alpha\\\vec\beta\end{bmatrix}(\tau)\\
&\quad\times\Big(\ddot{G}_B(\overline{\nu}_{12})(k_1\cdot\varepsilon_3\dot G_B(\overline{\nu}_{13}) + k_2\cdot\varepsilon_3\dot G_B(\overline{\nu}_{23}))\Big)\\
&\qquad+\lambda k_1\cdot k_2(\dot G_B(\overline{\nu}_{23})\varepsilon_3\cdot k_2 + \dot G_B(\overline{\nu}_{13})\varepsilon_3\cdot k_1)G_F\begin{bmatrix}\alpha_\uparrow\\\beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})^2\\
&\qquad+\lambda(\varepsilon_3\cdot k_1 k_2\cdot k_3 - \varepsilon_3\cdot k_2 k_1\cdot k_3)G_F\begin{bmatrix}\alpha_\uparrow\\\beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\\beta_\uparrow\end{bmatrix}(\overline{\nu}_{13})G_F\begin{bmatrix}\alpha_\uparrow\\\beta_\uparrow\end{bmatrix}(\overline{\nu}_{23})\Big)\\
&\quad\times G_F\begin{bmatrix}\alpha_G\\\beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\\beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\\beta_G\end{bmatrix}(\nu_{13})\\
&\quad\times\exp\Big(\lambda k_1\cdot k_2 G_B(\nu_{12}) + \lambda k_2\cdot k_3 G_B(\nu_{23}) + \lambda k_1\cdot k_3 G_B(\nu_{13})\Big)
\end{aligned} \tag{3.1}$$

where α_G and β_G are the boundary conditions on the torus of world sheet fermions associated with the gauge group of interest and $\nu_{ij} = \nu_i - \nu_j$. If the theory has unbroken space-time supersymmetries, the terms involving only bosonic Green functions vanish after summing over sectors because the partition function vanishes; in nonsupersymmetric theories, all terms are relevant. We can integrate by parts to remove the double derivative of the bosonic Green function, which gives (with an appropriate analytic continuation in the momentum invariants there are no boundary contributions from the contours around each of the other νs)

$$
\begin{aligned}
&-i\lambda\frac{4g^3}{16\pi^2}\,\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})\varepsilon_1\cdot\varepsilon_2\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu_1 d^2\nu_2\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta}C^{\vec\alpha}_{\vec\beta}Z\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau)\\
&\quad\Big[\dot G_B(\overline\nu_{12})\Big(\varepsilon_3\cdot k_1\dot G_B(\overline\nu_{13})\Big(k_1\cdot k_2\dot G_B(\overline\nu_{12})-k_2\cdot k_3\dot G_B(\overline\nu_{23})\Big)\\
&\qquad\qquad+\varepsilon_3\cdot k_2\dot G_B(\overline\nu_{23})\Big(k_1\cdot k_2\dot G_B(\overline\nu_{12})+k_1\cdot k_3\dot G_B(\overline\nu_{13})\Big)\Big)\\
&\quad-k_1\cdot k_2(\dot G_B(\overline\nu_{23})\varepsilon_3\cdot k_2+\dot G_B(\overline\nu_{13})\varepsilon_3\cdot k_1)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_{12})^2\\
&\quad-(\varepsilon_3\cdot k_1k_2\cdot k_3-\varepsilon_3\cdot k_2k_1\cdot k_3)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu_{23})\Big]\\
&\quad G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13})\\
&\quad\exp\Big(\lambda k_1\cdot k_2G_B(\nu_{12})+\lambda k_2\cdot k_3G_B(\nu_{23})+\lambda k_1\cdot k_3G_B(\nu_{13})\Big)
\end{aligned}
\tag{3.2}
$$

This term in the amplitude is proportional to the momentum invariant $k_i\cdot k_j$, which vanishes on-shell; it might thus appear that this contribution vanishes more strongly than the usual kinematic vanishing of the three-point amplitude. That is not right, however, because there are regions of punctured moduli space which produce poles in the momentum invariants, thereby yielding contributions to the β function [6,7].

Such apparent pole contributions† arise only in the regions of punctured moduli space where Koba-Nielsen variables are pinched together. Using the short distance behavior of the Green functions,

$$\dot G_B(\overline\nu) = -\frac{1}{2\pi\overline\nu}+O(\overline\nu)\qquad\qquad G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline\nu)=\frac{1}{2\pi\overline\nu}+O(\overline\nu)\tag{3.3}$$

the poles in the $k_i\cdot k_j$ channel arise from integrating over the ν_{ij} in the region $\nu_i\simeq\nu_j$. The only contributions are from integrals of the form

$$\int d^2\nu_i|\nu_{ij}|^{-2-\lambda k_i\cdot k_j/\pi} = -\frac{2\pi^2}{\lambda k_i\cdot k_j}\tag{3.4}$$

† In fact, because of the additional momentum factors in the numerator which cancel the pole, these apparent poles do *not* give rise to poles in the physical S-matrix; they are fake.

For example, to extract the $k_1 \cdot k_2$ pole from the $\varepsilon_1 \cdot \varepsilon_2$ terms we perform the integral around $\nu_{12} \simeq 0$ so that the relevant terms in the amplitude (3.2) reduce to

$$
\begin{aligned}
&-i\lambda\varepsilon_1 \cdot \varepsilon_2 \,\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3})\,\frac{4g^3}{16\pi^2}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu_{12}\, d^2\nu_{23}\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta} Z\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau)\\
&\qquad\times\left(\dot G_B(\overline{\nu}_{23})^2 - G_F\begin{bmatrix}\alpha_\Upsilon\\ \beta_\Upsilon\end{bmatrix}(\overline{\nu}_{23})^2\right)\\
&\qquad\times\frac{1}{(2\pi)^2}|\nu_{12}|^{-\lambda k_1\cdot k_2/\pi-2}\Big(-\varepsilon_3\cdot k_1\; k_2\cdot k_3 + \varepsilon_3 \cdot k_2\; k_1\cdot k_3\Big)\\
&\qquad\times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{23})\exp\Big(\lambda(k_2\cdot k_3 + k_1\cdot k_3)G_B(\nu_{23})\Big)\\
&= -i\varepsilon_1\cdot\varepsilon_2\,\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3})\,\frac{\varepsilon_3\cdot k_1\; k_2\cdot k_3 - \varepsilon_3\cdot k_2\; k_1\cdot k_3}{2k_1\cdot k_2}\\
&\qquad\times\frac{4g^3}{16\pi^2}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int_T d^2\nu\,(\mathrm{Im}\,\tau)\sum_{\vec\alpha,\vec\beta} C^{\vec\alpha}_{\vec\beta} Z\begin{bmatrix}\vec\alpha\\ \vec\beta\end{bmatrix}(\tau)\exp\left(\lambda(k_1+k_2)\cdot k_3 G_B(\nu)\right)\\
&\qquad\times\left(\dot G^2_B(\overline{\nu}) - G_F\begin{bmatrix}\alpha_\Upsilon\\ \beta_\Upsilon\end{bmatrix}(\overline{\nu})^2\right)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)
\end{aligned}
\tag{3.5}
$$

Collecting all the terms which contribute and using momentum conservation to replace $k_2 \cdot \varepsilon_3$ with $-k_1 \cdot \varepsilon_3$, we find that the kinematic factor associated with the $\varepsilon_1 \cdot \varepsilon_2$ term is

$$
\varepsilon_1\cdot\varepsilon_2\, k_1\cdot\varepsilon_3\left(\frac{k_2\cdot k_3 + k_1\cdot k_3}{k_1\cdot k_2} + (k_1\cdot k_2 + k_2\cdot k_3 + k_1\cdot k_3)\Big(\frac{1}{k_1\cdot k_3}+\frac{1}{k_2\cdot k_3}\Big)\right) \tag{3.6}
$$

which is ill-defined on shell, since both numerators and denominators vanish.

This difficulty persists for all N-point amplitudes; in the region of punctured moduli space where the loop is isolated on an external leg, the amplitude is ill-defined. In the case that the isolated leg is the last one, the denominator is

$$
\sum_{i<j<N} k_i\cdot k_j = k_N^2/2 = 0 \tag{3.7}
$$

while the numerator vanishes because of gauge invariance. (This configuration of Koba-Nielsen variables would correspond to a mass renormalization of the massless vector if the numerator did not vanish.)

Within the context of the Polyakov path integral Minahan [6] has proposed a prescription for dealing with this ambiguity. The prescription relaxes momentum conservation in such a way so that modular invariance is maintained. All external momenta and their sum $p = \sum_i k_i$ are required to be null vectors, so that

$$
\sum_{i<j}^{N} k_i\cdot k_j = 0 \qquad \text{and} \qquad k_i^2 = 0 \tag{3.8}
$$

In addition, $p \cdot \varepsilon$ is set to zero. Factorizing a four-point amplitude legitimates the use of this prescription for the three-point amplitude [9]; the kinematic invariant of equation (3.6) becomes simply

$$-\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 \tag{3.9}$$

4. Inconsistencies in the Covariant Loop Calculus

The ambiguity in Polyakov amplitudes displayed in the previous section has a painful echo in the covariant loop calculus, as we shall show in this section. In particular, for a general choice of the projective transformations V_i, the on-shell limit is ambiguous; and different N-point functions are not calculated consistently. In particular, if we look at the low-energy limit, the values of the various renormalization constants depend explicitly on the parameters of the projective transformations so that the β function, for example, is not well-defined†. In following sections, we shall see how the echo can be silenced by a suitable choice of the parameters in the projective transformations.

To simplify the discussion, in this section we shall retain only the terms relevant to a four-dimensional bosonic string. It is straightforward to include the additional superstring terms as given in the general amplitude (2.1); their inclusion does not eliminate any of the problems displayed in this section. In the covariant loop calculus, the three-vector amplitude for a four-dimensional closed bosonic string is (see Appendix I)

$$\begin{aligned}\mathcal{A}_3 = -\, i\frac{4g^3}{16\pi^2}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int_T d^2\nu_1 d^2\nu_2\,(\operatorname{Im}\tau)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\\ \times\prod_{i=1}^{3}\left[|d_i|^{\lambda k_i^2/\pi}e^{-\lambda k_i^2 \operatorname{Im}\nu_i}\right]\exp\Big(\lambda\sum_{i<j}^{3}k_i\cdot k_j\, G_B(\nu_{ij})\Big)R(1,2,3)L(1,2,3)\end{aligned} \tag{4.1}$$

where the right-mover contribution is

$$\begin{aligned}R(1,2,3) = \Big[-\varepsilon_1\cdot\varepsilon_2\,\ddot{G}_B(\overline{\nu}_{12})\Big(\varepsilon_3\cdot k_3\Big(ie^{-2\pi i\overline{\nu}_3}\bar{c}_3\bar{d}_3+\frac{i}{2}\Big)+\varepsilon_3\cdot k_1\dot{G}_B(\overline{\nu}_{13})+\varepsilon_3\cdot k_2\dot{G}_B(\overline{\nu}_{23})\Big)\\ +\text{cyclic}\Big]\\ -\lambda\Big(-\varepsilon_1\cdot k_1\Big(ie^{-2\pi i\overline{\nu}_1}\bar{c}_1\bar{d}_1+\frac{i}{2}\Big)+\varepsilon_1\cdot k_2\dot{G}_B(\overline{\nu}_{12})+\varepsilon_1\cdot k_3\dot{G}_B(\overline{\nu}_{13})\Big)\\ \times\Big(-\varepsilon_2\cdot k_2\Big(ie^{-2\pi i\overline{\nu}_2}\bar{c}_2\bar{d}_2+\frac{i}{2}\Big)-\varepsilon_2\cdot k_1\dot{G}_B(\overline{\nu}_{12})+\varepsilon_2\cdot k_3\dot{G}_B(\overline{\nu}_{23})\Big)\\ \times\Big(-\varepsilon_3\cdot k_3\Big(ie^{-2\pi i\overline{\nu}_3}\bar{c}_3\bar{d}_3+\frac{i}{2}\Big)-\varepsilon_3\cdot k_1\dot{G}_B(\overline{\nu}_{13})-\varepsilon_3\cdot k_2\dot{G}_B(\overline{\nu}_{23})\Big)\end{aligned} \tag{4.2}$$

† An analogous problem would arise in field theory if one attempted to calculate different N-point functions in different gauges; the β function computed from, say, Z_3 and Z_A would not agree with that computed from Z_4 and Z_A. In addition, physical Green functions would come out wrong, because the wavefunction renormalisation would have been calculated in a different gauge from the N-point Green function.

while the left-mover contribution is

$$\begin{aligned} L(1,2,3) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13}) \\ + \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{13}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{23}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{12}) \end{aligned} \tag{4.3}$$

Let us now make a simple choice of the parameters, $d_i = e^{-i\pi\nu_i}$, $\bar{d}_i = e^{i\pi\bar{\nu}_i}$, and $\bar{c}_i = -e^{2\pi i\bar{\nu}_i}/2\bar{d}_i$, and repeat the calculation of section 3. The details are quite similar, except that now the momenta are off-shell. Once again, after an integration by parts, the $\varepsilon_1 \cdot \varepsilon_2 \, \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})$ term is

$$\begin{aligned} \mathcal{A}_3 = & -i\lambda \frac{4g^3}{16\pi^2} \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})\, \varepsilon_1 \cdot \varepsilon_2 \\ & \times \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \int_T d^2\nu_1 d^2\nu_2 \, (\mathrm{Im}\,\tau) \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) \exp\Big(\lambda k_i \cdot k_j \sum_{i<j}^{3} G_B(\nu_{ij})\Big) \\ & \times \dot{G}_B(\bar{\nu}_{12}) \Big(\varepsilon_3 \cdot k_1 \dot{G}_B(\bar{\nu}_{13}) \Big(k_1 \cdot k_2 \dot{G}_B(\bar{\nu}_{12}) - k_2 \cdot k_3 \dot{G}_B(\bar{\nu}_{23})\Big) \\ & \qquad + \varepsilon_3 \cdot k_2 \dot{G}_B(\bar{\nu}_{23}) \Big(k_1 \cdot k_2 \dot{G}_B(\bar{\nu}_{12}) + k_1 \cdot k_3 \dot{G}_B(\bar{\nu}_{13})\Big)\Big) \\ & \times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23}) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{13}) \end{aligned} \tag{4.4}$$

This expression does not vanish since the $k_i \cdot k_j \neq 0$. Extracting the $k_1 \cdot k_2$ pole from the $\nu_1 \simeq \nu_2$ region, we find the ratio of momenta

$$\varepsilon_1 \cdot \varepsilon_2 \, \frac{-\varepsilon_3 \cdot k_1 k_2 \cdot k_3 + \varepsilon_3 \cdot k_2 k_1 \cdot k_3}{k_1 \cdot k_2} \tag{4.5}$$

The other terms yield similar results. Combining terms using momentum conservation, we arrive at an answer identical to equation (3.6), dropping terms proportional to $k_3 \cdot \varepsilon_3$:

$$\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 \Big(\frac{k_2 \cdot k_3 + k_1 \cdot k_3}{k_1 \cdot k_2} + (k_1 \cdot k_2 + k_2 \cdot k_3 + k_1 \cdot k_3)\Big(\frac{1}{k_1 \cdot k_3} + \frac{1}{k_2 \cdot k_3}\Big)\Big) \tag{4.6}$$

The ratio of momenta in the coefficient of $\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3$ is now quite well-defined; neither denominator nor numerator vanishes for general off-shell momenta. Unfortunately, the appearance of a non-trivial ratio of momenta is a disaster; it signals that the on-shell limit is an essential singularity of the theory. To see this explicitly, let us take the on-shell limit in two different ways. In both cases, take $k_3 \cdot \varepsilon_3 = 0$. First, let $k_3^2 = 0$, while k_1 and k_2 are still arbitrary; the kinematic invariant of equation (4.6) then becomes

$$\varepsilon_1 \cdot \varepsilon_2 \, k_1 \cdot \varepsilon_3 \Big(\frac{-k_3^2}{k_1 \cdot k_2} + (k_1 \cdot k_2 - k_3^2)\Big(\frac{-k_3^2}{k_1 \cdot k_3 \, k_2 \cdot k_3}\Big)\Big) = 0 \tag{4.7}$$

For the second limit, take $k_1^2 = k_2^2 = k_3^2 = s$. In the latter case, equation (4.6) becomes

$$\varepsilon_1 \cdot \varepsilon_2\, k_1 \cdot \varepsilon_3 \left(\frac{(-s)}{(-s/2)} - \frac{3}{2} s \frac{2}{(-s/2)} \right) = 8 \tag{4.8}$$

so that the on-shell limit is undefined. Like the case of on-shell Polyakov amplitudes discussed in the previous section, the difficulties here are not restricted to the three-point function; if we consider the contribution to the N-point function coming from a loop isolated on the N-th leg, we will find a momentum ratio

$$\frac{\sum_{i<N} k_i \cdot k_N}{\sum_{i<j<N} k_i \cdot k_j} = \frac{2k_N^2}{k_N^2 - \sum_{j<N} k_j^2} \tag{4.9}$$

This ratio also depends on the way the on-shell limit is taken.

It is clear that this disease cannot be cured by adding in contributions from regions of punctured moduli space where $N-1$ legs do not come together. In the three-point amplitude, these yield unambiguously vanishing contributions to the coefficient of $\varepsilon_1 \cdot \varepsilon_2\, k_1 \cdot \varepsilon_3$ in the on-shell limit, and thus cannot make the ill-defined ratio of momenta well-defined. For higher-point functions, the contributions of these other regions do not vanish identically, but *are* well-defined on-shell, and so cannot resolve the problem.

Another way of expressing the problem is to look at the limit of infinite string tension, where we expect to recover the loop amplitudes of a gauge theory which is the low energy effective field theory. Instead, we find amplitudes with pathological non-localities yielding an ill-defined S-matrix.

It is not enough to find a prescription that simply resolves the ratio of momenta to a pure number, however. We must ensure that resolution in different N-point functions is consistent. The problem is that different choices of the projective transformations yield different values for the renormalization constants. But the string theory has only one coupling constant, and there is only one kind of external field whose wavefunction renormalization enters into the amplitudes we are considering. Thus once we have calculated the values of two renormalization constants, say those of the physical three- and four-point amplitudes, we can determine the renormalization constants of all other physical amplitudes, in particular that of the two-point amplitude — the massless vector wavefunction renormalization. What happens if we look at the two-point amplitude itself? After an integration by parts to eliminate the $\ddot{G}_B$ terms, the bosonic terms in the two-point amplitude are

$$\mathcal{A}_2 = -\frac{1}{32\pi^2}(2g)^2\,\mathrm{Tr}(T^{a_1}T^{a_2})\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\left(\prod_{i=1}^{2}d\theta_{i3}d\theta_{i4}\right)\int_T d^2\nu_1(\mathrm{Im}\,\tau)\sum_{\vec{\alpha},\vec{\beta}}C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)$$
$$\times|e^{-\pi\,\mathrm{Im}\,\nu_1}d_1|^{\lambda k_1^2/\pi}|e^{-\pi\,\mathrm{Im}\,\nu_2}d_2|^{\lambda k_2^2/\pi}\exp\Big(\lambda k_1\cdot k_2 G_B(\nu_{12})\Big)$$
$$\times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{12})$$
$$\times\left(k_1\cdot k_2\varepsilon_1\cdot\varepsilon_2\;\dot{G}_B(\overline{\nu}_{12})\Big(\dot{G}_B(\overline{\nu}_{12})-\frac{\partial_{\overline{\nu}_1}|d_1|}{\pi|d_1|}-\frac{\partial_{\overline{\nu}_1}|d_2|}{\pi|d_2|}+\frac{i}{2}\Big)+\text{longitudinal}\right) \tag{4.10}$$

The value of the wavefunction renormalization is simply the coefficient of $\varepsilon_1\cdot\varepsilon_2$. This coefficient may be adjusted at will by choosing different projective transformations *without* modifying the other 'bare' N-point functions, thereby changing physical S-matrix elements, and spoiling the consistency of the results. (The β function provides an additional consistency check, as it should match known results for gauge-theory β-functions in the infinite-tension limit.)

Thus without making a special choice of the parameters of the projective invariances associated to each external leg in the candidate in the amplitude (2.1), the on-shell limit is ill-defined. Are there choices which make the on-shell limit well-defined, and yield consistent renormalization constants? There are; and in the following sections, we show how to make such choices.

5. Fixing the Projective Invariance

We seek a choice of the parameters of the projective transformations that will make the on-shell limit well-defined and gauge invariant, with consistent renormalization of different N-point amplitudes. Within the context of the string theory, we must also require the integrand to be well defined on the torus. It would also be nice to have world-sheet supersymmetry survive, so that spurious F_1 formalism states manifestly decouple off-shell as well as on-shell.

One such set of choices for the super projective transformation parameters satisfying the desired conditions is given by

$$|d_i| = e^{\pi\,\mathrm{Im}\,\nu_i}e^{\pi G_B(\nu_i-\nu_{i+1})/2} \tag{5.1a}$$
$$\bar{c}_i = i\frac{e^{2\pi i\overline{\nu}_i}}{\bar{d}_i}(\dot{G}_B(\overline{\nu}_i-\overline{\nu}_{i+1})+i/2) \tag{5.1b}$$
$$\bar{\gamma}_i = -\sqrt{-2\pi i}e^{i\pi\overline{\nu}_i}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_{i+1})\theta_{i+1,3}/\bar{d}_i \tag{5.1c}$$

where $\nu_{N+1}\equiv\nu_1$. The somewhat complicated appearance of this choice is an artifact of the use of the torus variables $\nu_i = \ln z_i/2\pi i$; as we shall see, it leads to a simple modification of the standard

Polyakov amplitude. We will explicitly demonstrate the suitability of this choice for the two-, three-, and four-point amplitudes. While this choice appears to attach significance to the labelling of the ν's, as we shall see in section 10, it is merely a simple example out of a family of choices that treat the Koba-Nielsen variables of the other external legs symmetrically.

With this choice of superprojective parameters the off-shell factors in the N massless-vector amplitude (2.1) are

$$
\begin{aligned}
&(e^{-2\pi i\nu_i}V_i'(0))^{-\lambda k_i^2/4\pi}(e^{i\pi\overline{\nu}_i}\overline{DV}_i^F(0))^{-\lambda k_i^2/2\pi} \\
&\qquad = |d_i|^{\lambda k_i^2/\pi}e^{-\lambda k_i^2\,\mathrm{Im}\,\nu_i}\left(1+i\frac{\lambda k_i^2}{2\pi}\sqrt{-2\pi i}e^{-i\pi\overline{\nu}_i}\bar{d}_i\theta_{i3}\bar{\gamma}_i/2\right) \\
&\qquad = \exp\left(\lambda\frac{k_i^2}{2}\left(G_B(\nu_i-\nu_{i+1})-\theta_{i3}\theta_{i+1,3}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_{i+1})\right)\right)
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
&\exp\left(-i\sqrt{\lambda}\theta_{i4}\varepsilon_i\cdot k_i\left(\frac{e^{-i\pi\overline{\nu}_i}}{\sqrt{-2\pi i}}\frac{\overline{D}^2\overline{V}_i^F(0)}{(\overline{DV}_i^F(0))^2}+i\frac{\theta_{i3}}{2}\right)\right) \\
&\qquad = \exp\left(-i\sqrt{\lambda}\theta_{i4}\varepsilon_i\cdot k_i\left(\frac{e^{-i\pi\overline{\nu}_i}}{\sqrt{-2\pi i}}\bar{d}_i(\bar{\gamma}_i+i\sqrt{-2\pi i}e^{-i\pi\overline{\nu}_i}\bar{c}_i\theta_{i3})+i\frac{\theta_{i3}}{2}\right)\right) \\
&\qquad = \exp\left(i\sqrt{\lambda}\theta_{i4}\varepsilon_i\cdot k_i\left(\theta_{i+1,3}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_{i+1})+\theta_{i3}\dot{G}_B(\overline{\nu}_i-\overline{\nu}_{i+1})\right)\right)
\end{aligned}
\tag{5.3}
$$

so that the N massless-vector scattering amplitude is

$$
\begin{aligned}
\mathcal{A}_N = {} & \frac{1}{2(16\pi^2)}\lambda^{N/2-2}(2g)^N\,T^{a_1}{}_{m_1}{}^{n_1}\cdots T^{a_N}{}_{m_N}{}^{n_N} \\
& \int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\left(\prod_{i=1}^{N}d\theta_{i1}\,d\theta_{i2}\,d\theta_{i3}\,d\theta_{i4}\right)\mathrm{Im}\,\tau\int_T\left(\prod_{i=1}^{N-1}d^2\nu_i\right)\sum_{\vec{\alpha},\vec{\beta}}C^{\vec{\alpha}}_{\vec{\beta}}\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) \\
& \prod_{i<j}\exp\Bigg[\lambda K_{ij}\,G_B(\nu_i-\nu_j)-\theta_{i3}\theta_{j3}\lambda K_{ij}\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad -\theta_{i1}\theta_{j2}\delta^{m_i}{}_{n_j}\,G_F\begin{bmatrix}\alpha_{m_i}\\ \beta_{n_j}\end{bmatrix}(\nu_i-\nu_j)-\theta_{i2}\theta_{j1}\delta^{m_j}{}_{n_i}\,\hat{G}_F\begin{bmatrix}\alpha_{m_j}\\ \beta_{n_i}\end{bmatrix}(\nu_i-\nu_j) \\
& \qquad +i\sqrt{\lambda}(\theta_{i3}\theta_{j4}E_{ji}+\theta_{i4}\theta_{j3}E_{ij})\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad -i\sqrt{\lambda}(\theta_{i3}\theta_{i4}E_{ij}-\theta_{j3}\theta_{j4}E_{ji})\,\dot{G}_B(\overline{\nu}_i-\overline{\nu}_j) \\
& \qquad +\theta_{i4}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\,G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i-\overline{\nu}_j)+\theta_{i3}\theta_{i4}\theta_{j3}\theta_{j4}\varepsilon_i\cdot\varepsilon_j\,\ddot{G}_B(\overline{\nu}_i-\overline{\nu}_j)\Bigg]
\end{aligned}
\tag{5.4}
$$

where $(i<j)$

$$
K_{ij}=\begin{cases}\delta_{i+1,j}k_i^2/2+k_i\cdot k_j\,, & j\neq N\\ \delta_{i,1}k_i^2/2+k_i\cdot k_j\,, & j=N\end{cases}
\tag{5.5}
$$

$$E_{ij} = \varepsilon_i \cdot (\delta_{i+1,j} k_i + k_j) \tag{5.6}$$

Since neither K_{ii} nor E_{ii} appear in the amplitude (5.4), we may define these to vanish. This off-shell amplitude is the same as the standard Polyakov one with the insertion of the factor

$$\prod_{i=1}^{N} \exp\left[\lambda \frac{k_i^2}{2} G_B(\nu_i - \nu_{i+1}) - \theta_{i3}\theta_{i+1,3}\lambda \frac{k_i^2}{2} G_F\begin{bmatrix}\alpha_\uparrow \\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i - \overline{\nu}_{i+1}) \right. \\ \left. + i\sqrt{\lambda}\theta_{i4}\theta_{i+1,3} k_i \cdot \varepsilon_i\, G_F\begin{bmatrix}\alpha_\uparrow \\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_i - \overline{\nu}_{i+1}) - i\sqrt{\lambda}\theta_{i3}\theta_{i4} k_i \cdot \varepsilon_i\, \dot{G}_B(\overline{\nu}_i - \overline{\nu}_{i+1})\right] \tag{5.7}$$

in the integrand. The K_{ij} and E_{ij} satisfy the important momentum conservation properties

$$\sum_{i<j}^{N} K_{ij} = 0 \tag{5.8}$$

$$\sum_{j=1,\neq i}^{N} E_{ij} = 0 \tag{5.9}$$

These are the basic properties which, as we shall show explicitly, resolve the on-shell ambiguity in the ratios of momenta. The first of these conditions is reminiscent of Minahan's condition (3.8).

Our choice also maintains the world sheet supersymmetry necessary for the decoupling of the spurious F_1 formalism states. This decoupling results from a cancellation between bosonic and fermionic Green functions. Since our choice consistently replaces $k_i \cdot k_j$ with K_{ij} and $\varepsilon_i \cdot k_j$ with E_{ij} throughout the amplitude, the decoupling of the spurious states is not affected. As a simple example of this decoupling consider the region where all Koba-Nielsen variables are close so that the loop is isolated on a tadpole. In this case, the leading singularity is due to the spurious F_1-formalism tachyon [16]. However, using the short distance behavior of the Green functions (3.3) and the correspondence of bosonic and fermionic Green functions (after integration by parts), the coefficient of this singularity vanishes manifestly and the spurious tachyon decouples.

What is more striking is that the amplitude (5.4) is also modular invariant. This follows from the modular transformation properties of the bosonic and fermionic Green functions given in Appendix IV. The invariance under $\tau \to \tau + 1$ is straightforward, and the Grassman integrations make the invariance of the factors involving fermionic Green functions and derivatives of the bosonic Green functions under $\tau \to -1/\tau$ easy to demonstrate as well. This leaves us with the question of the invariance of the exponential of the bosonic Green functions; with our choice the potential violation appears as a factor $\prod_i |\tau|^{\lambda k_i^2/2\pi} \prod_{i<j} |\tau|^{\lambda k_i \cdot k_j/\pi}$ which is unity because of momentum conservation. Without the additional off-shell factors of equation (5.7) the amplitude would not, of course, be off-shell modular invariant.

On-shell, modular invariance is necessary for unitarity. Off-shell, it is nice, because it makes the restriction of the τ integral in equation (5.4) to the fundamental region a natural, rather than an ad hoc, prescription. It is also natural, since it is an off-shell extension of an on-shell symmetry. (We stress that unlike Giddings, Martinec, and Witten [17], or Kaku and Lykken [18], we do not mean 'off-shell modular invariance' merely in the sense of covering the modular region properly, or of regaining modular invariance in the on-shell limit; we mean honest-to-goodness modular invariance of off-shell amplitudes.) While we shall not prove any stronger statements, we suspect that its significance is much deeper, that modular invariance off-shell is *necessary* for the existence of a well-defined on-shell limit. This suggests that a consistent string field theory must maintain modular invariance off-shell as well as on-shell.

6. Resolution of the Inconsistency in the Three-point Amplitude

Let us first see how our choice of projective transformations resolves the unwanted ratio of momenta in the three-point amplitude. After performing the Grassman integrations, and then integrating by parts, the amplitude (5.4) becomes

$$
\begin{aligned}
\mathcal{A}_3 = &-i\lambda\frac{4g^3}{16\pi^2}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\operatorname{Im}\tau)^2}\int_T d^2\nu_1 d^2\nu_2(\operatorname{Im}\tau)\sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\\
&\times R(1,2,3)L(1,2,3)\exp\Big(\lambda K_{12}G_B(\nu_{12})+\lambda K_{23}G_B(\nu_{23})+\lambda K_{13}G_B(\nu_{13})\Big)
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
R(1,2,3) = \Big[&\varepsilon_1\cdot\varepsilon_2\dot{G}_B(\overline{\nu}_{12})\Big(E_{31}\dot{G}_B(\overline{\nu}_{13})\Big(K_{12}\dot{G}_B(\overline{\nu}_{12})-K_{23}\dot{G}_B(\overline{\nu}_{23})\Big)\\
&\qquad+E_{32}\dot{G}_B(\overline{\nu}_{23})\Big(K_{12}\dot{G}_B(\overline{\nu}_{12})+K_{13}\dot{G}_B(\overline{\nu}_{13})\Big)\Big)\\
&-\varepsilon_1\cdot\varepsilon_2K_{12}\Big(\dot{G}_B(\overline{\nu}_{23})E_{32}+\dot{G}_B(\overline{\nu}_{13})E_{31}\Big)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})^2\\
&-\varepsilon_1\cdot\varepsilon_2\Big(E_{31}K_{23}-E_{32}K_{13}\Big)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{23})+\text{cyclic}\Big]\\
&\qquad-\Big(E_{12}\dot{G}_B(\overline{\nu}_{12})+E_{13}\dot{G}_B(\overline{\nu}_{13})\Big)\Big(E_{23}\dot{G}_B(\overline{\nu}_{23})-E_{21}\dot{G}_B(\overline{\nu}_{12})\Big)\\
&\qquad\times\Big(-E_{31}\dot{G}_B(\overline{\nu}_{13})-E_{32}\dot{G}_B(\overline{\nu}_{23})\Big)\\
&\qquad+\Big(E_{12}E_{21}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})^2(E_{31}\dot{G}_B(\overline{\nu}_{13})+E_{32}\dot{G}_B(\overline{\nu}_{23}))+\text{cyclic}\Big)\\
&\qquad+\Big(E_{12}E_{23}E_{31}-E_{13}E_{21}E_{32}\Big)G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{23})
\end{aligned}
\tag{6.2}
$$

and $L(1,2,3)$ is the same as in equation (4.3).

After using the momentum conservation conditions (5.8) and (5.9), R simplifies to

$$
\begin{aligned}
R(1,2,3) = \Bigg[&\varepsilon_1 \cdot \varepsilon_2 \; E_{32} K_{12} \Big(\dot{G}_B(\overline{\nu}_{12}) \Big(-\dot{G}_B(\overline{\nu}_{13}) \dot{G}_B(\overline{\nu}_{12}) + \dot{G}_B(\overline{\nu}_{13}) \dot{G}_B(\overline{\nu}_{23}) \\
&\qquad + \dot{G}_B(\overline{\nu}_{23}) \dot{G}_B(\overline{\nu}_{12}) \Big) + G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{12}) G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{13}) G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{23}) \\
&\qquad + (\dot{G}_B(\overline{\nu}_{23}) - \dot{G}_B(\overline{\nu}_{13})) G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{12})^2 \Big) + \text{cyclic} \Bigg] \\
&+ E_{13} E_{21} E_{32} \Bigg[\Big(\dot{G}_B(\overline{\nu}_{13}) - \dot{G}_B(\overline{\nu}_{12}) \Big) \Big(\dot{G}_B(\overline{\nu}_{23}) + \dot{G}_B(\overline{\nu}_{12}) \Big) \Big(\dot{G}_B(\overline{\nu}_{13}) - \dot{G}_B(\overline{\nu}_{23}) \Big) \\
&\qquad + \Big(G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{12})^2 (\dot{G}_B(\overline{\nu}_{13}) - \dot{G}_B(\overline{\nu}_{23})) + \text{cyclic} \Big) \\
&\qquad - 2 G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{12}) G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{13}) G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu}_{23}) \Bigg]
\end{aligned} \tag{6.3}
$$

As discussed in section 4, the problems in the amplitude occur in the regions where the loop is isolated on an external leg. There are three such regions corresponding to the three pinchings of the Koba-Nielsen variables. The $(\varepsilon \cdot k)^3$ terms do not contribute in these regions because the coefficients of such potential contributions vanish. This is as expected, since otherwise the massless-vector amplitude would be divergent in the on-shell limit. After extracting the various pole terms, one finds

$$
\begin{aligned}
\mathcal{A}_3 = &- i \frac{2g^3}{16\pi^2} \operatorname{Tr}(T^{a_1}[T^{a_2}, T^{a_3}])(\varepsilon_1 \cdot \varepsilon_2 \; \varepsilon_3 \cdot k_2 + \text{cyclic}) \\
&\times \int_{\mathcal{F}_1} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \int_T d^2\nu (\operatorname{Im}\tau) \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\, e^{\lambda(K_{23}+K_{13})G_B(\nu)} \\
&\times \Big(\dot{G}_B^2(\overline{\nu}) - G_F\begin{bmatrix}\alpha_\dagger\\ \beta_\dagger\end{bmatrix}(\overline{\nu})^2 \Big) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)
\end{aligned} \tag{6.4}
$$

so that the on-shell ambiguity has disappeared. The reader will recognize this as an equivalent of the usual Yang-Mills three-point vertex multiplied by a loop factor.

7. Resolution of the Inconsistency in the Four-Point Amplitude

We shall now show explicitly how our choice of parameters (5.1a–5.1c) resolves the ambiguities in the four-point amplitude. We shall focus on one particular term, but the behavior is similar for all terms in the amplitude (indeed, the infinite-tension limit is once again identical to a calculation performed elsewhere [9]). The analysis of the four-point amplitude is a nontrivial demonstration of the consistency of our choice of projective transformations, since $\log(\lambda)$ terms from pinches of different Koba-Nielsen variables must combine to form an object that on shell is gauge-invariant and proportional to the tree amplitude.

As our example we choose the $\epsilon_1 \cdot \epsilon_2\ \epsilon_3 \cdot \epsilon_4\ \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ term given by

$$
\begin{aligned}
\mathcal{T}_4 = & \frac{\lambda^2}{4}\frac{2^3 g^4}{(16\pi^2)}\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\,\mathrm{Im}\,\tau\int_T d^2\nu_1\, d^2\nu_2\, d^2\nu_3 \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\exp\Big(\lambda\sum_{i<j}^{4} K_{ij}G_B(\nu_{ij})\Big)\\
& \times\Bigg[\dot G_B(\overline{\nu}_{12})\dot G_B(\overline{\nu}_{34})\Big(K_{12}K_{34}\dot G_B(\overline{\nu}_{12})\dot G_B(\overline{\nu}_{34}) + K_{34}K_{14}\dot G_B(\overline{\nu}_{34})\dot G_B(\overline{\nu}_{14})\\
& \quad -K_{12}K_{23}\dot G_B(\overline{\nu}_{12})\dot G_B(\overline{\nu}_{23}) - K_{14}K_{23}\dot G_B(\overline{\nu}_{14})\dot G_B(\overline{\nu}_{23}) + K_{34}K_{13}\dot G_B(\overline{\nu}_{13})\dot G_B(\overline{\nu}_{34})\\
& \quad -K_{12}K_{13}\dot G_B(\overline{\nu}_{13})\dot G_B(\overline{\nu}_{12}) + K_{13}K_{24}\dot G_B(\overline{\nu}_{13})\dot G_B(\overline{\nu}_{24})\Big)\\
& \quad - K_{34}\dot G_B(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{34})^2\Big(K_{12}\dot G_B(\overline{\nu}_{12}) + K_{14}\dot G_B(\overline{\nu}_{14}) + K_{13}\dot G_B(\overline{\nu}_{13})\Big)\\
& \quad - K_{12}\dot G_B(\overline{\nu}_{34})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})^2\Big(K_{34}\dot G_B(\overline{\nu}_{34}) - K_{23}\dot G_B(\overline{\nu}_{23}) - K_{13}\dot G_B(\overline{\nu}_{13})\Big)\\
& \quad +G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{34})\Bigg(K_{12}K_{34}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{34})\\
& \qquad - K_{13}K_{24}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{13})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{24})\\
& \qquad + K_{14}K_{23}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{14})G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{23})\Bigg)\Bigg]\\
& \quad \times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{23})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{34})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{14})
\end{aligned}
\tag{7.1}
$$

where the momentum factors K_{ij} are defined in equation (5.8) and we have integrated by parts so as to eliminate $\ddot{G}_B$ terms.

Let us examine the massless-vector pole that arises when the loop is isolated on the last leg. (Higher-mass states do not contribute, since they will not yield an on-shell propagator.) To extract this pole, it is convenient to change coördinates to

$$
\begin{aligned}
\nu &= \nu_{34}\\
\eta &= \nu_{13}\\
\omega &= \nu_{12}/\eta
\end{aligned}
\tag{7.2}
$$

The η coordinate is then the size of a (small) disk on the torus which contains ν_1, ν_2, and ν_3, while ν gives the location of this disk, and ω the relative locations of ν_1 and ν_2 within the disk.

The pole arises from the $\eta \simeq 0$ region, from terms which have a factor of $|\eta|^{-2-\lambda K_{ij}/\pi}$. Terms with mismatched powers of η and $\overline{\eta}$ will be killed by the integration over the phase of η, while terms with a higher negative power of $|\eta|$ correspond to the would-be propagation of the fictitious tachyon and will cancel by virtue of world-sheet supersymmetry. In the $\eta \to 0$ limit, the Green functions behave as follows,

$$
\begin{aligned}
G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{i4}) &\to G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}) + \mathcal{O}(\overline{\eta})\\
G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{13}) &\to \frac{1}{\overline{\eta}} G_F^{\mathrm{tree}}(1) + \mathcal{O}(\overline{\eta})\\
G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{23}) &\to \frac{1}{\overline{\eta}} G_F^{\mathrm{tree}}(1-\overline{\omega}) + \mathcal{O}(\overline{\eta})\\
G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu}_{12}) &\to \frac{1}{\overline{\eta}} G_F^{\mathrm{tree}}(\overline{\omega}) + \mathcal{O}(\overline{\eta})\\
\dot{G}_B(\overline{\nu}_{i4}) &\to \dot{G}_B(\overline{\nu}) + \mathcal{O}(\overline{\eta})\\
\dot{G}_B(\overline{\nu}_{13}) &\to \frac{1}{\overline{\eta}} \dot{G}_B^{\mathrm{tree}}(1) + \mathcal{O}(\overline{\eta})\\
\dot{G}_B(\overline{\nu}_{23}) &\to \frac{1}{\overline{\eta}} \dot{G}_B^{\mathrm{tree}}(1-\overline{\omega}) + \mathcal{O}(\overline{\eta})\\
\dot{G}_B(\overline{\nu}_{12}) &\to \frac{1}{\overline{\eta}} \dot{G}_B^{\mathrm{tree}}(\overline{\omega}) + \mathcal{O}(\overline{\eta})
\end{aligned}
\tag{7.3}
$$

with analogous formulæ for the left-movers.

Looking at the region where $\eta \simeq 0$, and extracting only terms with appropriate powers of $|\eta|$, one finds that equation (7.1) becomes

$$
\begin{aligned}
\mathcal{T}_4 =& \frac{\lambda^2}{4}\frac{2^3 g^4}{(16\pi^2)} \int d^2\eta |\eta|^{-2-\lambda(K_{12}+K_{23}+K_{13})/\pi} \\
&\times \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\,\mathrm{Im}\,\tau \int_T d^2\nu \int d^2\omega \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) \exp\Big(\lambda \sum_{i<4} K_{i4} G_B(\nu)\Big) \\
&\quad \times \exp\Big(\lambda K_{12} G_B^{\rm tree}(\omega) + \lambda K_{23} G_B^{\rm tree}(1-\omega)\Big) \\
&\times \Bigg[\Big(\dot{G}_B(\overline{\nu})^2 - G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu})^2\Big) \Big(K_{12}K_{34}\left(\dot{G}_B^{\rm tree}(\overline{\omega})^2 - G_F^{\rm tree}(\overline{\omega})^2\right) \\
&\quad + K_{34}K_{13}\dot{G}_B^{\rm tree}(\overline{\omega})\dot{G}_B^{\rm tree}(1)\Big) \\
&\quad + \dot{G}_B(\overline{\nu})^2 \left(-K_{14}K_{23}\dot{G}_B^{\rm tree}(\overline{\omega})\dot{G}_B^{\rm tree}(1-\overline{\omega}) + K_{13}K_{24}\dot{G}_B^{\rm tree}(\overline{\omega})\dot{G}_B^{\rm tree}(1)\right) \\
&\quad + G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu})^2 \left(K_{14}K_{23}G_F^{\rm tree}(\overline{\omega})G_F^{\rm tree}(1-\overline{\omega}) - K_{13}K_{24}G_F^{\rm tree}(\overline{\omega})G_F^{\rm tree}(1)\right) \\
&\quad - \lambda K_{12}\dot{G}_B(\overline{\nu})^2 \left(K_{23}\dot{G}_B^{\rm tree}(1-\overline{\omega}) + K_{13}\dot{G}_B^{\rm tree}(1)\right) \\
&\qquad\qquad \times \left(K_{14} + (1-\overline{\omega})K_{24}\right)\left(\dot{G}_B^{\rm tree}(\overline{\omega})^2 - G_F^{\rm tree}(\overline{\omega})^2\right)\Bigg] \\
&\times G_F^{\rm tree}(-\omega)G_F^{\rm tree}(-(1-\omega))G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu)
\end{aligned}
\tag{7.4}
$$

The term with the additional explicit power of λ arises from the expansion in $\overline{\eta}$ of the exponentiated bosonic Green functions.

Putting in the explicit form of the tree-level Green functions, we find

$$
\begin{aligned}
\mathcal{T}_4 =& \frac{\lambda^2}{4}\frac{2g^4}{(16\pi^2)2\pi^2} \int d^2\eta |\eta|^{-2-\lambda(K_{12}+K_{23}+K_{13})/\pi} \\
&\quad \times \int_{\mathcal{F}_1} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\,\mathrm{Im}\,\tau \int_T d^2\nu \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) \exp\Big(\lambda \sum_{i<4} K_{i4} G_B(\nu)\Big) \\
&\quad\quad \times \left(G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu})^2 - \dot{G}_B(\overline{\nu})^2\right) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu)G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu) \\
&\times \left(K_{14}K_{23}\mathcal{I}_1(K_{12},K_{23}) - K_{13}\left(K_{34}+K_{24}\right)\mathcal{I}_2(K_{12},K_{23})\right)
\end{aligned}
\tag{7.5}
$$

where

$$
\begin{aligned}
\mathcal{I}_1(s,t) &= \frac{1}{2\pi^2}\int d^2\omega\, |\omega|^{-2-\lambda s/\pi}|1-\omega|^{-2-\lambda t/\pi} \\
\mathcal{I}_2(s,t) &= \frac{1}{2\pi^2}\int d^2\omega\, |\omega|^{-2-\lambda s/\pi}|1-\omega|^{-\lambda t/\pi}(1-\omega)^{-1}
\end{aligned}
\tag{7.6}
$$

Evaluating the integrals gives

$$\begin{aligned}\mathcal{I}_1(s,t) &= -\left(\frac{1}{\lambda s}+\frac{1}{\lambda t}\right)\, J(s,t)\\ \mathcal{I}_2(s,t) &= -\frac{1}{\lambda s}\, J(s,t)\end{aligned} \tag{7.7}$$

where

$$J(s,t) = \frac{\Gamma(1-\lambda s/2\pi)\Gamma(1-\lambda t/2\pi)\Gamma(1+\lambda(s+t)/2\pi)}{\Gamma(1+\lambda s/2\pi)\Gamma(1+\lambda t/2\pi)\Gamma(1-\lambda(s+t)/2\pi)} \tag{7.8}$$

Evaluating the η integral and substituting these expressions for the $\mathcal{I}_i$ reduces equation (7.5) to

$$\begin{aligned}\mathcal{T}_4 =& \frac{2g^4}{4(16\pi^2)} \int_{\mathcal{F}_1} \frac{d^2\tau}{(\operatorname{Im}\tau)^2} \operatorname{Im}\tau \int_T d^2\nu \sum_{\vec{\alpha},\vec{\beta}} C^{\vec{\alpha}}_{\vec{\beta}} Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) \exp\Big(\lambda \sum_{i<4} K_{i4} G_B(\nu)\Big)\\ &\times \left(G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\overline{\nu})^2 - \dot{G}_B(\overline{\nu})^2\right)\, G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu) G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu)\\ &\times \frac{J(K_{12},K_{23})}{K_{12}+K_{13}+K_{23}}\, \left(K_{14}K_{23}\left(\frac{1}{K_{12}}+\frac{1}{K_{23}}\right) - K_{13}\left(K_{34}+K_{24}\right)\frac{1}{K_{12}}\right)\end{aligned} \tag{7.9}$$

Using the condition (5.8) we can eliminate K_{24} and rewrite the ratio of momentum invariants,

$$\begin{aligned}&\frac{1}{K_{12}+K_{13}+K_{23}}\, \left(K_{14}K_{23}\left(\frac{1}{K_{12}}+\frac{1}{K_{23}}\right) - K_{13}\left(K_{34}+K_{24}\right)\frac{1}{K_{12}}\right)\\ &\quad= \frac{1}{K_{12}+K_{13}+K_{23}}\, \left(K_{14}\frac{K_{23}+K_{12}+K_{13}}{K_{12}} + K_{13}\frac{K_{12}+K_{13}+K_{23}}{K_{12}}\right)\\ &\quad= \frac{K_{14}+K_{13}}{K_{12}}\end{aligned} \tag{7.10}$$

The last expression has a well-defined on-shell limit.

As noted at the beginning of this section, the calculation in the low-energy limit is identical to one performed using the Minahan prescription, and the improper non-localities disappear in *all* terms, in a manner consistent with their resolution in the three-point case. We will not present the details here.

One may also calculate renormalization constants using the results of the previous and current sections. Such a computation is essentially identical to one done with Polyakov amplitudes and Minahan's 'off-sheet' prescription, so we shall not repeat it. The results of the latter computations [9] show that the combination of the three- and four-point renormalization constants determine that the wavefunction renormalization of massless vectors vanishes at one loop. One may also extract a β-function from these renormalization constants. In the infinite tension limit, it turns out to be identical to the β-function derived directly from the appropriate limiting gauge field theory, as of course it must for consistency of the results.

8. Resolution of the Inconsistency in the N-Point Amplitude

In this section, we give a general argument that our choice of superprojective parameters (5.1a–5.1c) resolves the improper non-localities in all N-point massless-vector amplitudes. The basic idea is that integration by parts can extract the appropriate momentum factors for bosonic terms, while world-sheet supersymmetry extends the resolution to fermionic terms. Integration by parts simply adds total derivatives to the integrand, and does not change the full amplitude. It is sufficient to demonstrate the resolution of the improper non-localities for a particular choice of additional total derivative terms.

In the one-loop N massless-vector amplitude (2.1) we have fixed the last Koba-Nielsen variable to a particular value. Thus, we cannot integrate by parts with respect to it, something that would be convenient to do for the purposes of this section. This is easy to solve: simply fix a different Koba-Nielsen variable, or integrate over all variables but divide by the volume of the torus ($\mathrm{Im}\,\tau$).

Let us consider the region of moduli space where the loop is isolated on the N-th leg. To extract the massless-vector pole, it is convenient to make the following change of variables,

$$
\begin{aligned}
\nu &= \nu_{2N} \\
\eta &= \nu_{12} \\
\omega_i &= \nu_{i,i+1}/\eta \qquad i = 3, 4, \ldots, N-1
\end{aligned}
\tag{8.1}
$$

We will take ν_1 to be the fixed Koba-Nielsen variable. Once again, η is the size of a disk on the torus, near ν_1, which contains the points $\nu_1, \ldots, \nu_{N-1}$, while ν is the relative location of the the N-th vertex operator, and ω_i are the relative locations of the points within the disk.

Under this change of variables the measure becomes

$$
\int \prod_{i=2}^{N} d^2\nu_i = \int d^2\nu\, d^2\eta \prod_{i=3}^{N-1} d^2\omega_i\, |\eta|^{2N-6}
\tag{8.2}
$$

In the region $\eta \simeq 0$ only contributions of the form

$$
\int d^2\eta\, |\eta|^{-2} \prod_{i<j<N}^{N} |\eta|^{-\lambda K_{ij}/\pi} = -\frac{2\pi^2}{\lambda \sum_{i<j<N}^{N} K_{ij}}
\tag{8.3}
$$

will yield a massless-vector pole on the last leg. Just as in the case of the four-point amplitude, terms with other powers of η and $\overline{\eta}$ will either die or fail to contribute poles. It is thus a simple matter of expanding the integrand in the region $\eta \simeq 0$, and counting powers of η and $\overline{\eta}$ in order to find the potential pole contributions; the right-mover Green functions (not including the exponentiated ones) must contribute $2-N$ powers of $\overline{\eta}$ while the left-mover Green functions must contribute $2-N$ powers of η. This means that the right-movers must have either one $\ddot{G}_B(\overline{\nu}_{jN})$, or two $\dot{G}_B(\overline{\nu}_{jN})$, or two $G_F[{}^{\alpha_\Upsilon}_{\beta_\Upsilon}](\overline{\nu}_{jN})$, since otherwise the remaining bosonic or fermionic Green functions ($\dot{G}_B(\overline{\nu}_{ij})$, $G_F[{}^{\alpha_\Upsilon}_{\beta_\Upsilon}](\overline{\nu}_{ij})$, or $\tilde{G}_B(\overline{\nu}_{ij})$ where $i, j \neq N$) would contribute the wrong number of powers of $\overline{\eta}^{-1}$.

Let us concentrate on those terms where ν_N appears only in the argument to bosonic Green functions. Here it is convenient to examine the amplitude (5.4) *before* integrating by parts. Each term is of course linear in ε_N; let us consider in turn those terms where this polarization vector is dotted into another polarization vector, and those terms where it is dotted into a momentum vector.

The first of these two sub-types has the form

$$\varepsilon_j \cdot \varepsilon_N \, \tilde{G}_B(\overline{\nu}_{jN}) \times (\text{other factors}) \tag{8.4}$$

As discussed above, if the other factors contain $\overline{\nu}_N$ dependence then this term will not lead to a massless-vector pole on the external leg, and there will be no on-shell ambiguity. But this makes it easy to integrate by parts with respect to $\overline{\nu}_N$. Integrating by parts turns the $\tilde{G}_B$ into a $\dot{G}_B$, and brings down a factor from the exponentiated Green functions. The right-mover term (8.4) then becomes

$$-\varepsilon_j \cdot \varepsilon_N \dot{G}_B(\overline{\nu}_{jN}) \sum_{i=1}^{N-1} K_{iN} \dot{G}_B(\overline{\nu}_{iN}) \times (\text{other factors}) \tag{8.5}$$

In the limit $\eta \to 0$, the bosonic Green functions simplify:

$$\dot{G}_B(\overline{\nu}_{iN}) \to \dot{G}_B(\overline{\nu}) + \mathcal{O}(\overline{\eta}) \tag{8.6}$$

Doing the η integral thus gives us the pole contribution,

$$\frac{\sum_{j=1}^{N-1} K_{jN}}{\sum_{i<j<N}^{N} K_{ij}} = -1 \tag{8.7}$$

where we have used the momentum conservation condition (5.8).

For the second sub-type, where ε_N is contracted into one of the k_j the form of the amplitude is

$$\sum_{j=i}^{N-1} E_{Nj} \dot{G}_B(\overline{\nu}_{jN}) \times (\text{ other factors}) \tag{8.8}$$

In this case the pole piece is

$$\frac{\sum_{j=i}^{N-1} E_{Nj}}{\sum_{i<j<N}^{N} K_{ij}} \tag{8.9}$$

which vanishes using equation (5.9).

Therefore, the ambiguity disappears from terms which contain bosonic Green functions of $\overline{\nu}_N$ (but arbitrary mixtures of fermionic and bosonic Green functions of the other variables). What about fermionic Green functions of $\bar{\nu}_N$? Here we may appeal to world sheet supersymmetry, which guarantees that the bosonic and fermionic contributions to poles of spurious F_1-formalism states cancel. There is, after all, one more pinch that we can perform: we can bring ν_N close to all the other νs. This potentially leads to poles in the new throat variable η' representing the contribution of the spurious F_1-formalism tachyon. World-sheet supersymmetry, which is preserved by our choice (5.1a–5.1c), assures us that we can add total derivatives to the amplitude so that the offending terms have the form

$$G_F^{\text{tree}}(\eta')^2\ \left(\dot{G}_B^{\text{tree}}(\bar{\eta}')^2 - G_F^{\text{tree}}(\bar{\eta}')^2\right) \times \text{ other factors} \tag{8.10}$$

so that the spurious state disappears explicitly. For this cancellation to occur consistently in all terms in the amplitude the kinematic structure of the terms containing fermionic Green functions of ν_N must be identical to the terms with bosonic Green functions of ν_N. In particular, any on-shell ambiguities must have disappeared from the terms with fermionic Green functions, leaving behind the same constants as in the bosonic case. We have seen this fermionic-bosonic matching explicitly in the form of the three- and four-point amplitudes after integration by parts, shown respectively in equations (6.1) and (7.1).

The form of the amplitude is similar at higher loops, and thus the choices analogous to equations (5.1a–5.1c) will resolve ambiguities there in a similar fashion.

9. Consistency in the Two-Point Amplitude

In the two-point Polyakov amplitude, there is only one momentum invariant; and in any event, no pinch is needed to isolate the loop on an external leg, since it is already there. So in this case there will be no on-shell momentum/pole ambiguity.

Nonetheless, there is a consistency requirement. As discussed in sections 5 and 7, the wavefunction renormalization extracted from the two-point amplitude must match that deduced from the combination of the three- and four-point amplitudes; that is, it must vanish. As we shall show, this requirement, along with the requirement that the spurious F_1-formalism tachyon decouple off-shell, fixes the parameters of the projective transformations to be of the form given by equations (5.1a–5.1c).

How do we adjust the parameters of the projective transformations so the wavefunction renormalization vanishes? For the two-point function the number of possibile choices for the θ_{iJ} dependence of the $\bar{\gamma}_i$ is quite small. It is not sensible to have the $\bar{\gamma}_i$ depend on the θ_{i4} since these are parameters which were introduced by hand to select the terms multi-linear in the polarization vectors. It is also not difficult to show that the only possible choice which can yield a vanishing wavefunction renormalization is $\bar{\gamma}_1 \propto \theta_{23}$ and $\bar{\gamma}_2 \propto \theta_{13}$.

Taking $\bar{\gamma}_1 = \bar{g}_1\theta_{23}$ and $\bar{\gamma}_2 = \bar{g}_2\theta_{13}$, where the g_i are arbitrary c-number parameters, and substituting into the general covariant loop calculus amplitude (2.1) yields the two-point amplitude (see also Appendix II)

$$\begin{aligned}
\mathcal{A}_2 =& \frac{1}{32\pi^2}\lambda^{-1}(2g)^2\,\mathrm{Tr}(T^{a_1}T^{a_2})\int_{\mathcal{F}_1}\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}\int\left(\prod_{i=1}^{2}d\theta_{i3}d\theta_{i4}\right)\int_T d^2\nu_1(\mathrm{Im}\,\tau)\sum_{\vec{\alpha},\vec{\beta}}C^{\vec{\alpha}}_{\vec{\beta}}Z\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau)\\
&\times|e^{-\pi\,\mathrm{Im}\,\nu_1}d_1|^{\lambda k_1^2/\pi}|e^{-\pi\,\mathrm{Im}\,\nu_2}d_2|^{\lambda k_2^2/\pi}\exp\Big(\lambda k_1\cdot k_2 G_B(\nu_{12})\Big)\\
&\Bigg[\varepsilon_1\cdot\varepsilon_2\,\Bigg(\ddot{G}_B(\bar{\nu}_{12})+G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{12})\\
&\times\Bigg(\lambda k_1\cdot k_2 G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{12})-i\lambda k_1^2\sqrt{-2\pi i}e^{-i\pi\bar{\nu}_1}\bar{d}_1\bar{g}_1/4\pi+i\lambda k_2^2\sqrt{-2\pi i}e^{-i\pi\bar{\nu}_2}\bar{d}_2\bar{g}_2/4\pi\Bigg)\Bigg)\\
&\quad-\lambda\varepsilon_1\cdot k_1\,\varepsilon_2\cdot k_2\Big(\dot{G}_B(\bar{\nu}_{12})+(i\bar{c}_1\bar{d}_1e^{-2\pi i\bar{\nu}_1}+i/2)\Big)\Big(\dot{G}_B(\bar{\nu}_{21})+(i\bar{c}_2\bar{d}_2e^{-2\pi i\bar{\nu}_2}+i/2)\Big)\\
&\quad-\lambda\varepsilon_1\cdot k_1\,\varepsilon_2\cdot k_2\Big(G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{12})+\frac{e^{-\pi i\bar{\nu}_1}}{\sqrt{-2\pi i}}\bar{d}_1\bar{g}_1\Big)\Big(G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{21})+\frac{e^{-\pi i\bar{\nu}_2}}{\sqrt{-2\pi i}}\bar{d}_2\bar{g}_2\Big)\Bigg]\\
&\times G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(-\nu_{12})G_F\begin{bmatrix}\alpha_G\\ \beta_G\end{bmatrix}(\nu_{12})
\end{aligned} \tag{9.1}$$

The wavefunction renormalization is entirely determined by the coefficient of the $\varepsilon_1\cdot\varepsilon_2$ term, while the renormalization of the "gauge-fixing parameter" depends also on the longitudinal terms. The former coefficient must vanish no matter what the string model, and so the fermionic and bosonic contributions must vanish independently.

Since two of $k_1\cdot k_2$, k_1^2, and k_2^2 are independent, requiring the fermionic contributions to vanish tells us that

$$\bar{g}_1 = -\sqrt{-2\pi i}e^{i\pi\bar{\nu}_1}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{12})/\bar{d}_1\,,\qquad \bar{g}_2 = \sqrt{-2\pi i}e^{i\pi\bar{\nu}_2}G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{12})/\bar{d}_2 \tag{9.2}$$

while requiring the bosonic contributions to vanish tells us that the bosonic integrand,

$$|e^{-\pi\,\mathrm{Im}\,\nu_1}d_1|^{\lambda k_1^2/\pi}|e^{-\pi\,\mathrm{Im}\,\nu_2}d_2|^{\lambda k_2^2/\pi}\exp\Big(\lambda k_1\cdot k_2 G_B(\nu_{12})\Big)\ddot{G}_B(\bar{\nu}_{12}) \tag{9.3}$$

must be a total derivative in $\bar{\nu}_1$ or $\bar{\nu}_2$. This can happen only if the factors other than $\ddot{G}_B(\bar{\nu}_{12})$ combine to yield a constant; taking the logarithm of these factors and using momentum conservation, we have the equation

$$\lambda k_1^2/\pi\log|e^{-\pi\,\mathrm{Im}\,\nu_1}d_1|+\lambda k_2^2/\pi\log|e^{-\pi\,\mathrm{Im}\,\nu_2}d_2|+\lambda k_1\cdot k_2 G_B(\nu_{12}) = \mathrm{constant} \tag{9.4}$$

which has the solution

$$|d_1| = \mathrm{constant}\times e^{\pi\,\mathrm{Im}\,\nu_1}\exp(\pi G_B(\nu_{12})/2)\,,\qquad |d_2| = \mathrm{constant}\times e^{\pi\,\mathrm{Im}\,\nu_2}\exp(\pi G_B(\nu_{12})/2) \tag{9.5}$$

The constant factors merely result in factors of $(\text{constant})^{-k_i^2}$ in front of the amplitude; they are irrelevant and we shall drop them.

The astute reader will have noticed that this does not quite make the bosonic terms a total derivative. The problem is that there is a 'surface term' because the derivative of the $G_F[{}^{\alpha a}_{\beta a}](\nu)$ with respect to $\overline{\nu}$ is not zero but is rather a delta function, as the Green function has a simple pole at $\nu = 0$. Having fished this out, however, we note that it is a røde sild: it is a contact term of the Green-Seiberg type [19]. In the superstring case, these terms are an artifact of the spurious F_1 formalism tachyon and can be eliminated by an appropriate analytic continuation in momenta. For a bosonic string the tachyon is physical, but again an analytic continuation eliminates the contact terms. To make the analytic continuation, integrate by parts to eliminate the $\tilde{G}_B$ *before* fixing the choice (9.5); with an appropriate choice of external momenta there are no surface terms. At the end, when the choice (9.5) is substituted the $\varepsilon_1 \cdot \varepsilon_2$ term in the amplitude vanishes identically. This technicality does not arise in higher-point functions, since there the momentum invariants are in general non-vanishing off-shell, and the leading uncancelled contribution from the left-movers is not ν^{-1} but rather $\nu^{-1-\lambda K_{ij}/2\pi}$.

While we are disposing of technicalities, we may also note that the region where the two Koba-Nielsen variables come together, yielding a dilaton tadpole, gives no real difficulties. In a superstring model with any space-time supersymmetries (or an Atkin-Lehner symmetry [20]), this tadpole vanishes identically.

It is amusing that with the choice (9.2), the fermionic Green function contributions to the longitudinal terms vanish automatically. Imposing the condition that the spurious tachyon of the F_1 formalism drops out of the longitudinal parts of the amplitude means that we must also choose

$$\bar{c}_1 = i\frac{e^{2\pi i\overline{\nu}_1}}{\bar{d}_1}(\dot{G}_B(\overline{\nu}_{12}) + i/2) \qquad \bar{c}_2 = i\frac{e^{2\pi i\overline{\nu}_2}}{\bar{d}_2}(\dot{G}_B(\overline{\nu}_{21}) + i/2) \tag{9.6}$$

which also makes the contributions from the bosonic Green functions vanish in the longitudinal terms. (A different choice of the $\bar{c}_i$ will not alter the wavefunction renormalization but will lead to a nonvanishing renormalization of the longitudinal pieces.)

The choice of superprojective transformation parameters (9.5–9.6) is exactly that given by equations (5.1a–5.1c), which as shown in previous sections also resolves the inconsistencies in the three- and four-point amplitudes.

10. A Bundle of Projective Transformations

As we saw in sections 6–8, the momentum conservation conditions (5.8) and (5.9) sufficed to resolve the on-shell ambiguity in any amplitude of the form given by equation (5.4), where the

Green functions are the same as those appearing in the standard Polyakov amplitude, but their coefficients are a bit different.

For the case of the bosonic string, a more general set of choices of the projective transformation parameters that preserves the momentum conservation conditions is given by

$$|d_i| = \eta_i e^{\pi \,\mathrm{Im}\, \nu_i} \exp\Big(\pi \sum_{j<k}^{N} a^i_{jk} G_B(\overline{\nu}_{jk})/2\Big) \tag{10.1}$$

$$\bar{c}_i = i\frac{e^{2\pi i \overline{\nu}_i}}{\bar{d}_i}\Big(\sum_{j=1}^{N} b_{ij}\dot{G}_B(\overline{\nu}_{ij}) + \frac{i}{2}\Big) \tag{10.2}$$

where the η_i are arbitrary constants, and the coefficients a^i_{jk} and b_{ij} satisfy

$$\sum_{j<k}^{N} a^i_{jk} = 1\ , \qquad a^i_{jj} = 0 \tag{10.3}$$

and

$$\sum_{j}^{N} b_{ij} = 1\ , \qquad b_{ii} = 0 \tag{10.4}$$

In this case, the momentum invariants are

$$K_{ij} = k_i \cdot k_j + \sum_{l=1}^{N} a^l_{ij} k_l^2/2 \tag{10.5}$$

$$E_{ij} = \varepsilon_i \cdot k_j + b_{ij}\varepsilon_i \cdot k_i. \tag{10.6}$$

which do indeed satisfy the fundamental conditions (5.8) and (5.9). The $\bar{c}_i$, which determine the form of the longitudinal $\varepsilon_i \cdot k_i$ terms, must be associated with Green functions of the form $\dot{G}_B(\overline{\nu}_{ij})$; otherwise these terms could not be combined with the nonvanishing on-shell terms to form the E_{ij}. For the k_i^2 pieces there is no analogous restriction and the $|d_i|$ can depend on all $G_B(\nu_{jk})$.

In the superstring case, the requirement that the $\varepsilon_i \cdot k_i$ terms combine with the $\varepsilon_i \cdot k_j$ to form the E_{ij} forces the $\bar{\gamma}_i$ to depend only on the $G_F[{}^{\alpha_\dagger}_{\beta_\dagger}](\overline{\nu}_{ij})$. The requirement that the spurious F_1 formalism states decouple off-shell demands a cancellation between bosonic and fermionic terms that in turn relates the $|d_i|$ and $\bar{c}_i$ to the $\bar{\gamma}_i$ Here, a more general set of super-projective transformation parameters is

$$|d_i| = \eta_i e^{\pi \,\mathrm{Im}\, \nu_i} \exp\Big(\pi \sum_{j=1}^{N} a_{ij} G_B(\nu_{ij})/2\Big) \tag{10.7}$$

$$\bar{c}_i = i\frac{e^{2\pi i \overline{\nu}_i}}{\bar{d}_i}\Big(\sum_{j=1}^{N} a_{ij}\dot{G}_B(\overline{\nu}_{ij}) + i/2\Big) \tag{10.8}$$

$$\bar{\gamma}_i = -\sqrt{-2\pi i}\,\frac{e^{i\pi\bar{\nu}_i}}{\bar{d}_i}\sum_{j=1}^{N} a_{ij} G_F\begin{bmatrix}\alpha_\uparrow\\ \beta_\uparrow\end{bmatrix}(\bar{\nu}_{ij})\theta_{j3} \tag{10.9}$$

where the coefficients a_{ij} satisfy

$$\sum_{j=1}^{N} a_{ij} = 1\ , \qquad a_{ii} = 0 \tag{10.10}$$

(The requirement of cancellation of the spurious F_1 formalism states is responsible for the appearance of the a_{ij} in all three paramters.) Such a choice of superprojective parameters yields an amplitude of the form given in equation (5.4) but with

$$K_{ij} = k_i \cdot k_j + a_{ij}k_i^2/2 + a_{ji}k_j^2/2 \tag{10.11}$$

and

$$E_{ij} = \varepsilon_i \cdot k_j + a_{ij}\varepsilon_i \cdot k_i \tag{10.12}$$

which also satisfy the fundamental conditions (5.8) and (5.9). We do not know if these choices are the most general ones possible, but the calculation of the two-point amplitude in section 9 suggests they may well be.

From amongst the choices described in this section, the crossing symmetric one (all a_{ij} equal) is the one that would likely emerge from a consistent covariant closed string field theory.

What about higher loops? We conjecture that the prescription is the same as at one loop, namely, taking the standard Polyakov amplitude and replacing $k_i \cdot k_j$ with K_{ij} and $\varepsilon_i \cdot k_j$ with E_{ij}, where the K_{ij} and E_{ij} are chosen from the sets described in this section. This yields the higher loop analogs of equation (5.4). It is worth noting that at tree level, this prescription yields amplitudes invariant under the same projective transformations that leave the on-shell amplitudes invariant.

11. Conclusions

In this paper, we have presented an off-shell string scattering amplitude for massless-vector external states. This amplitude may be viewed as a special case of the general amplitude in the covariant loop calculus, but unlike the latter, our amplitude possesses a well-defined on-shell limit. Likewise, it gives a well-defined meaning to the Polyakov amplitude in those regions of punctured moduli space where the latter is ill-defined.

Our off-shell amplitude gives a consistent set of renormalization constants for amplitudes with different numbers of external legs. This requirement may seem trivial to a field theorist, but as we have seen, it is in fact quite non-trivial for amplitudes in a first-quantized formalism. As a further consistency check, the infinite-tension limit yields the correct β-function for the limiting field theory. We have shown explicitly the resolution of the ambiguities and the consistency of the renormalization constants in the two-, three-, and four-point amplitudes; and we have given an argument for the resolution of the ambiguities for all N-point amplitudes.

While we have considered only massless-vector external states in this paper, we expect similar ambiguities and similar resolutions for any gauge particle, such as the graviton. It may also be possible to extend the ideas presented here to the massive states of the string.

The consistency requirements satisfied by the off-shell amplitudes presented in this paper must also be satisfied by amplitudes arising out of any string field theory: the latter must resolve the on-shell ambiguities of Polyakov amplitudes in a manner consistent with gauge invariance. For 'string field' formalisms not directly based on an underlying string-field action, the consistency requirement on renormalization constants is also a non-trivial constraint. The unexpected fact that our amplitudes are fully modular-invariant *off*-shell may give an important clue to the structure of a consistent string field theory. Moreover, it may be possible to construct a string field action from the off-shell amplitudes considered in this work.

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Appendix I. Bosonic String Covariant Loop Calculus.

In this appendix we briefly review the bosonic string covariant loop calculus and present the open bosonic string massless-vector amplitudes with emphasis on their projective invariance.

One of the key observations of this paper is that the projective invariance inherent in the N-string vertex can be used to make off-shell amplitudes well defined. The projective invariance of the amplitudes was well-known in the early days of string theory [21]. Among the variety of choices that have appeared in the literature, the Lovelace choice (which has the advantage of manifest cyclic symmetry of the vertex and is defined in terms of only the Koba-Nielsen variables) is the preferred choice in the operator approach of Di Vecchia et al. [3].

In the path integral approach of Petersen et al. [4], the projective transformations arise naturally from the sewing process and are left as a manifest invariance of the amplitudes; there is no need to specify the detailed form of the V_i's. In this approach the three-vertices are used as basic building blocks which are sewn together to form the amplitudes. One sews by integrating the physical and ghost fields living on the punctured Riemann surface in question; boundary conditions at a given puncture are given by 'boundary' fields that are singular at that puncture and contain information about the external string state corresponding to the puncture. These 'boundary' fields are then written as 'standard' fields, singular at $z = 0$, acted on by some projective transformation $V_i(z)$ that moves the singularity from $z = 0$ to the real puncture at $z = z_i$. That is, the V_i must satisfy the condition (2.5). This does not totally fix V_i (which has 3 degrees of freedom).

For our purposes it is important to choose the V_i's to depend on the Teichmuller parameters. In the path integral approach of ref. [4] the vertex is always constructed for fixed but arbitrary values of the moduli; dependence on moduli which arise in future sewings is no problem. Equivalently, it is a simple matter to re-arrange the sewing process so that all the loops are first sewn from the fundamental three-vertices into "tadpoles". Then the various three-vertices and tadpoles are sewn together to form the N-point amplitude. This re-ordering corresponds simply to rearranging the order of the path integrals. Once a three-vertex is sewn into a tadpole the modular parameter is a well defined quantity which can then enter into the V_i's.

In the operator form of the covariant loop calculus, each asymptotic string state in an N-string scattering is represented by its own distinct Fock space. The N-string g-loop vertex is then simply a vector $\langle V_{N,g}|$ in the direct product space, with the property that the matrix element

$$\langle V_{N,g}| \bigotimes_{i=1}^{N} |\Phi_i\rangle \tag{I.1}$$

is the amplitude for scattering of the N string states $|\Phi_i\rangle$.

After eliminating the ghost degrees of freedom (and ignoring the Chan-Paton factors), the N-string vertex can be written in the compact form [3, 22]

$$\langle V_{N,g}| = \int d\mathcal{V} \langle V_{N,0}| L \tag{I.2}$$

where $\langle V_{N,0}|$ is the N-string tree-vertex. The latter may be written as

$$\begin{aligned}\langle V_{N,0}| = &\int \prod_{i=1}^{N} \left(\frac{dz_i \theta(z_i - z_{i+1})}{V_i'(0)} \right) \frac{1}{d\mathcal{V}_{abc}} \prod_{i=1}^{N} {}_i\langle\, x = 0, 0_a | \\ &\exp \left[\sum_{i<j} \sum_{n,m=0}^{\infty} \frac{\alpha_n^i}{n!} \partial_z^n \partial_y^m \ln \left[V_i(z) - V_j(y) \right] \Bigg|_{z=y=0} \frac{\alpha_m^j}{m!} \right] \\ &\exp \left[\frac{1}{2} \sum_{i=1}^{N} \sum_{n=0}^{\infty} \sqrt{2\alpha'} k_i \frac{\alpha_n^i}{n!} \partial_z^n \ln V_i'(z) \Bigg|_{z=0} \right]\end{aligned} \tag{I.3}$$

where L is the loop-correction factor

$$L = \prod_{i,j=1}^{N} \exp \left[\frac{1}{2} \sum_{n,m=0}^{\infty} \frac{\alpha_n^i}{n!} \partial_z^n \partial_y^m N\left(V_i(z), V_j(y)\right) \Bigg|_{z=y=0} \frac{\alpha_m^j}{m!} \right] . \tag{I.4}$$

The α_n^i's are the oscillator operators of the i-th string Hilbert space, normalized in the standard fashion

$$\left[(\alpha_n^i)^\dagger, \alpha_m^i \right] = -n \delta_{n,m} \ , \quad n, m > 0 \ ; \qquad\qquad \alpha_0^i = \sqrt{2\alpha'} k_i \ , \quad n = 0 \ . \tag{I.5}$$

In the general case the function $N(z, y)$ is quite complicated [3], but for $g = 1$ a considerable simplification occurs. If we use overall projective invariance to choose the fixed points of the Schottky generator as $\eta = \infty$ and $\zeta = 0$, we may write

$$\begin{aligned} N(z, y) &= \ln \left[\prod_{n=1}^{\infty} \frac{(z - q^n y)(y - q^n z)}{zy(1 - q^n)^2} \right] + \frac{1}{2 \ln q} \left(\ln \frac{z}{y} \right)^2 \\ &= \tilde{G}_B(z, y) - \ln |z - y| \end{aligned} \tag{I.6}$$

where $q = e^{2\pi i \tau}$. We may also choose $z_a = \eta$, $z_b = \zeta$ in $d\mathcal{V}_{abc}$, thereby obtaining the integration measure for the Teichmüller parameter

$$\frac{d\mathcal{V}}{d\mathcal{V}_{abc}} = \frac{dq}{q^2} \frac{z_c}{dz_c} \left(\frac{-2\pi}{\ln q} \right)^{d/2} \prod_{n=1}^{\infty} (1 - q^n)^{-d+2} \tag{I.7}$$

where we still have the freedom to fix one of the Koba-Nielsen variables to be z_c.

The one-loop amplitude with external massless vectors is obtained by saturating the N-string vertex (I.2) with N massless-vector states. Introducing Grassmann parameters which select those terms that are multi-linear in the polarization vectors, the amplitude can be written as (up to normalization)

$$\mathcal{M} = \int \prod_{i=1}^{N} d\theta_{i2} d\theta_{i1} \int \frac{dq}{q^2} \frac{z_c}{dz_c} \prod_{n=1}^{\infty} (1-q^n)^{-d+2} \left(\frac{-2\pi}{\ln q}\right)^{d/2}$$
$$\prod_{i=1}^{N} \left[\int dz_i\, \theta(z_i - z_{i+1})\, (V_i'(0))^{\alpha' k_i^2} \exp\Big(\frac{1}{2}\sqrt{2\alpha'}\theta_{i1}\theta_{i2} k_i \cdot \varepsilon_i \frac{V_i''(0)}{(V_i'(0))^2}\Big)\right]$$
$$\prod_{i\neq j} \exp\Big(\alpha' k_i \cdot k_j \tilde{G}_B(z_i, z_j) + \theta_{j1}\theta_{j2}\sqrt{2\alpha'} k_i \cdot \varepsilon_j \partial_{z_j} \tilde{G}_B(z_i, z_j)$$
$$+ \frac{1}{2}\theta_{i1}\theta_{i2}\theta_{j1}\theta_{j2}\varepsilon_i \cdot \varepsilon_j \partial_{z_i}\partial_{z_j} \tilde{G}_B(z_i, z_j)\Big) \tag{I.8}$$

where we have used $N(z,z) = \partial_z N(z,z) = 0$.

Converting to the annulus variables $\nu_i = \ln z_i/2\pi i$ and $\tau = \ln q/2\pi i$ the amplitude is

$$\mathcal{M} = \int \prod_{i=1}^{N} d\theta_{i2} d\theta_{i1} \int d\tau \frac{1}{d\nu_c} e^{-2\pi i \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})^{-d+2} (-i\tau)^{-d/2}$$
$$\prod_{i=1}^{N} \left[\int d\nu_i\, (e^{-2\pi i \nu_i} V_i'(0))^{\alpha' k_i^2} \exp\Big(\sqrt{2\alpha'}\theta_{i1}\theta_{i2} k_i \cdot \varepsilon_i\, i\pi \Big(e^{2\pi i \nu_i} \frac{V_i''(0)}{(V_i'(0))^2} - 1\Big)\Big)\right] \tag{I.9}$$
$$\prod_{i\neq j} \exp\Big(\alpha' k_i \cdot k_j \tilde{G}_B(\nu_{ij}) - \theta_{j1}\theta_{j2}\sqrt{2\alpha'} k_i \cdot \varepsilon_j \dot{\tilde{G}}_B(\nu_{ij})$$
$$- \frac{1}{2}\theta_{i1}\theta_{i2}\theta_{j1}\theta_{j2}\varepsilon_i \cdot \varepsilon_j \ddot{\tilde{G}}_B(\nu_{ij})\Big)$$

where

$$\tilde{G}_B(\nu_{ij}) = \tilde{G}_B(z_i, z_j) - \frac{1}{2}\ln(z_i z_j) \tag{I.10}$$

and $\dot{\tilde{G}}_B(\nu) \equiv \partial_\nu \tilde{G}_B(\nu)$.

From the above expression we may, for example, easily calculate the two-gluon amplitude

$$\mathcal{M}_2 = \int d\tau \int d\nu_1\, e^{-2\pi i \tau} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})^{-d+2} (-i\tau)^{-d/2} (e^{-2\pi i \nu_1} V_1'(0))^{\alpha' k_1^2} (e^{-2\pi i \nu_2} V_2'(0))^{\alpha' k_2^2}$$
$$\exp\Big(2\alpha' k_1 \cdot k_2 \tilde{G}_B(\nu_{12})\Big)\Big(\varepsilon_1 \cdot \varepsilon_2 \ddot{\tilde{G}}_B(\nu_{12}) - 2\alpha' \varepsilon_1 \cdot k_1\, \varepsilon_2 \cdot k_2 \Big(\dot{\tilde{G}}_B(\nu_{12})$$
$$\times \Big(\dot{\tilde{G}}_B(\nu_{12}) - i\pi e^{2\pi i \nu_1} \frac{V_1''(0)}{(V_1'(0))^2} + i\pi\Big) \Big(\dot{\tilde{G}}_B(\nu_{21}) - i\pi e^{2\pi i \nu_2} \frac{V_2''(0)}{(V_2'(0))^2} + i\pi\Big)\Big) \tag{I.11}$$

where we have used momentum conservation and ν_2 is fixed.

The essential difference between the bosonic left-movers of the heterotic models used in the text and the standard $d = 26$ open bosonic string discussed here is that some coordinates are internal and have been fermionized. (In addition, there are the usual closed string modifications of the zero-mode contributions to the Green functions.) These internal coordinates are not relevant to the extension of Polyakov amplitudes to off-shell quantities since off-shellness is a property of the external momenta. Thus, the off-shell amplitudes for the left-movers of a four-dimensional heterotic string may be determined using the vertex (I.2) for external oscillators and the usual on-shell results for the internal oscillators.

Appendix II. Superstring Covariant Loop Calculus

It is the superstring covariant loop calculus that is more relevant to this paper, since the right-movers of the heterotic string considered in the text may as usual be considered as an open superstring (except for minor differences in the bosonic zero-mode contributions), and it is amongst the right-movers that all the interesting action happens. Fortunately, for those sectors where there are no technical difficulties arising from fermionic zero modes, the formalism is similar to the bosonic string case.

After eliminating the ghost degrees of freedom, the N-superstring vertex in the Neveu-Schwarz sector can be expressed in the compact form [3,4]

$$\langle V_{N,g}| = \int d\mathcal{V} \langle V_{N,0}| L \tag{II.1}$$

where $\langle V_{N,0}|$ is the N-string tree-amplitude [23] which may be written as [3,4]

$$\begin{aligned}\langle V_{N,0}| = \int \prod_{i=1}^{N} \left(\frac{dZ_i}{DV_i^F(0)} \right) \frac{1}{d\mathcal{V}_{abc}} \prod_{i=1}^{N} {}_i\langle x = 0, 0_a| \\ \exp\left[\sum_{i<j} \sum_{n,m=0}^{\infty} \frac{A_n^i}{[n]!} D_Z^{2n} \frac{A_m^j}{[m]!} D_Y^{2m} \ln\left[V_i(Z) - V_j(Y)\right] \Big|_{Z=Y=0} \right] \\ \exp\left[\sum_{i=1}^{N} \sum_{n=0}^{\infty} \sqrt{2\alpha'} k_i A_n^i \frac{1}{[n]!} D_Z^{2n} \ln DV_i^F(Z) \Big|_{Z=0} \right]\end{aligned} \tag{II.2}$$

The loop-correction factor is

$$L = \prod_{i,j=1}^{N} \exp\left[\frac{1}{2} \sum_{n,m=0}^{\infty} \frac{A_n^i}{[n]!} D_Z^{2n} \frac{A_m^j}{[m]!} D_Y^{2m} \mathcal{N}(V_i(Z), V_j(Y)) \Big|_{Z=Y=0} \right] \tag{II.3}$$

The n and m run over both the integers and the half-integers with

$$[n] = \begin{cases} n \text{ for } n \text{ integer} \\ n - \frac{1}{2} \text{ for } n \text{ half odd integer} \end{cases} \tag{II.4}$$

The A_n^i's are the oscillator operators of the i'th string Hilbert space

$$
\begin{aligned}
A_n^i &= \alpha_n^i \qquad & [\alpha_n^i, \alpha_m^i] &= n\delta_{n+m,0} \qquad & n, m &= \pm 1, \pm 2, \dots \\
A_n^i &= \psi_n^i \qquad & [\psi_n^i, \psi_m^i] &= \delta_{n+m,0} \qquad & n, m &= \pm\frac{1}{2}, \pm\frac{3}{2}, \dots
\end{aligned}
\tag{II.5}
$$

with $\alpha_0^i = \sqrt{2\alpha'}k_i$. The superprojective transformations are constrained by the condition (2.6). In these formulæ we used superspace notation; if $Z = (z, \theta)$ and $Y = (y, \zeta)$ are two points in superspace then

$$
Z - Y \equiv z - y - \theta\zeta \tag{II.6}
$$

and $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ is the superderivative satisfying the supersymmetry algebra $D^2 = \partial$.

Just as in the bosonic case, the function $\mathcal{N}(Z, Y)$ appearing is in general quite complicated [3,4], but for $g = 1$ a considerable simplification occurs. If we use overall superprojective invariance to choose $U = (\infty, 0)$ and $T = (0, 0)$ as the fixed points of the super Schottky-generator, for the annulus

$$
\begin{aligned}
\mathcal{N}(Z, Y) &= \ln\left[\prod_{n=1}^{\infty} \frac{(z - q^n y - \theta\zeta q^{n/2})(y - q^n z - \zeta\theta q^{n/2})}{zy(1 - q^n)^2}\right] + \frac{1}{2\ln q}\left(\ln\frac{z}{y}\right)^2 \\
&= G(Z, Y) - \ln(Z - Y) \\
&= \tilde{G}_B(z, y) - \ln(z - y) + \theta\zeta\tilde{G}_F(z, y) + \frac{\theta\zeta}{z - y}
\end{aligned}
\tag{II.7}
$$

with $q = e^{2\pi i\tau}$. The two possible spin structures correspond to the two signs one can choose for $q^{1/2}$ in the $d = 10$ superstring.

We may also choose $Z_a = U$, $Z_b = T$ in $d\mathcal{V}_{abc}$, thereby obtaining the integration measure for the Teichmüller parameter of the annulus

$$
\frac{d\mathcal{V}}{d\mathcal{V}_{abc}} = \frac{dq}{q^{3/2}}\frac{z_c}{dz_c}\left(\frac{-2\pi}{\ln q}\right)^{d/2}\prod_{n=1}^{\infty}\left(\frac{1 - q^{n-1/2}}{1 - q^n}\right)^{d-2} \tag{II.8}
$$

and once again, we still have the freedom to fix one of the Koba-Nielsen variables to be z_c.

Saturating the N-string vertex with N massless-vector states, the contribution to the Euclidean amplitude is

$$
\begin{aligned}
\mathcal{M} = &\int \frac{dq}{q^{3/2}}\frac{z_c}{dz_c}\prod_{n=1}^{\infty}\left(\frac{1 - q^{n-1/2}}{1 - q^n}\right)^{d-2}\left(\frac{-2\pi}{\ln q}\right)^{d/2} \\
&\prod_{i=1}^{N}\left[\int dz_i d\theta_i d\phi_i (DV_i^F(0))^{2\alpha' k_i^2}\exp\left[\sqrt{2\alpha'}\phi_i k_i\cdot\varepsilon_i\frac{D^2V_i^F(0)}{(DV_i^F(0))^2}\right]\right] \\
&\prod_{i\neq j}\exp\Big[\alpha' k_i\cdot k_j G(Z_i, Z_j) + \sqrt{2\alpha'}\phi_j k_i\cdot\varepsilon_j D_{Z_j}G(Z_i, Z_j) \\
&\qquad - \frac{1}{2}\phi_i\phi_j\varepsilon_i\cdot\varepsilon_j\, D_{Z_i}D_{Z_j}G(Z_i, Z_j)\Big]
\end{aligned}
\tag{II.9}
$$

where we have introduced auxiliary Grassmann variables ϕ_i which select the terms multi-linear in the polarization vectors. In obtaining this amplitude we used the superconformal condition

$$DV_i^B(Z) = V_i^F(Z)DV_i^F(Z) \tag{II.10}$$

as well as $\mathcal{N}(Z,Z) = D\mathcal{N}(Z,Z) = 0$ and $(\psi^i_{1/2})^2 = 0$.

By expanding the superfields, changing to the annulus variables ν_i and τ and performing the rescalings

$$\theta_i \to i\sqrt{2\pi i}e^{\pi i\nu_i}\theta_i \qquad \phi_i \to -i\sqrt{2\pi i}e^{\pi i\nu_i}\phi_i \qquad \varepsilon_i \to i\varepsilon_i \tag{II.11}$$

we obtain the form of the right-movers' contribution to the amplitude (2.1) given in the text

$$\begin{aligned}\mathcal{M} = &\int d\tau \frac{1}{d\nu_c} e^{-i\pi\tau} \prod_{n=1}^{\infty}\left(\frac{1-e^{2\pi i(n-1/2)\tau}}{1-e^{2\pi i n\tau}}\right)^{d-2}\left(\frac{-2\pi}{\ln q}\right)^{d/2} \\ &\prod_{i=1}^{N}\left[\int d\nu_i d\theta_i d\phi_i\left(\frac{e^{-\pi i\nu_i}}{d_i} + i\frac{\sqrt{2\pi i}}{2}\theta_i\gamma_i\right)^{2\alpha' k_i^2}\right. \\ &\qquad \left.\exp\left[2\pi i\sqrt{2\alpha'}\phi_i k_i\cdot\varepsilon_i\left(e^{\pi i\nu_i}d_i\left(\frac{\gamma_i}{\sqrt{2\pi i}} - ie^{i\pi\nu_i}c_i\theta_i\right) - i\frac{\theta_i}{2}\right)\right]\right] \\ &\prod_{i<j}\exp\left[2\alpha' k_i\cdot k_j(\tilde{G}_B(\nu_{ij}) - \theta_i\theta_j\tilde{G}_F(\nu_{ij}))\right. \\ &+ i\sqrt{2\alpha'}(\theta_i\phi_j k_i\cdot\varepsilon_j + \phi_i\theta_j k_j\cdot\varepsilon_i)\tilde{G}_F(\nu_{ij}) - i\sqrt{2\alpha'}(\theta_i\phi_i k_j\cdot\varepsilon_i - \theta_j\phi_j k_i\cdot\varepsilon_j)\dot{\tilde{G}}_B(\nu_{ij}) \\ &\left.+\varepsilon_i\cdot\varepsilon_j\phi_i\phi_j\tilde{G}_F(\nu_{ij}) + \varepsilon_i\cdot\varepsilon_j\theta_i\phi_i\theta_j\phi_j\ddot{\tilde{G}}_B(\nu_{ij})\right]\end{aligned} \tag{II.12}$$

where c_i, d_i, γ_i are free parameters in the superprojective transformation (2.3), $\tilde{G}_B(\nu_{ij})$ is defined in eq. (I.10) and $\tilde{G}_F(\nu_{ij}) \equiv 2\pi i\sqrt{z_i z_j}\tilde{G}_F(z_i,z_j)$. The rescaling of the θ_i and ϕ_i by $\sqrt{2\pi i}e^{i\pi\nu_i}$ is a simple way to include the Jacobian arising from the change of variables from the z_i to the $\nu_i = \ln z_i/2\pi i$, while the factors of $\pm i$ arise from the choice of conventions used in the text. Note that if we take the γ_i to depend on the θ_{i+1} as in sect. 3 then we must include an extra factor of $i\sqrt{2\pi i}e^{i\pi\nu_{i+1}}$ arising from the rescaling (II.11). (Of course, this factor is not important since we can always absorb it into the γ_i.) An overall complex conjugation (not including the $i\varepsilon$) as well as a rotation to Minkowski space must be performed in order to obtain the formulae of the text for the right-movers. Note also that the open string Green functions of the appendices are normalized differently then the closed string Green functions of the text.

From the general amplitude (II.9)we may, for example, easily calculate the two-gluon amplitude

$$\mathcal{M}_2 = \int \frac{dq}{q^{3/2}} \frac{dZ_1 dZ_2 z_c}{dz_c} \prod_{n=1}^{\infty} \left(\frac{1-q^{n-1/2}}{1-q^n} \right)^{d-2} \left(\frac{-2\pi}{\ln q} \right)^{d/2} (DV_1^F(0))^{2\alpha' k_1^2} (DV_2^F(0))^{2\alpha' k_2^2}$$
$$\exp\left(2\alpha' k_1 \cdot k_2\, G(Z_1, Z_2)\right) \Bigg[\epsilon_1 \cdot \epsilon_2 D_{Z_1} D_{Z_2} G(Z_1, Z_2)$$
$$+ 2\alpha' \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \left(\frac{D^2 V_1^F(0)}{(DV_1^F(0))^2} - D_{Z_1} G(Z_1, Z_2) \right) \left(\frac{D^2 V_2^F(0)}{(DV_2^F(0))^2} - D_{Z_2} G(Z_1, Z_2) \right) \Bigg] \tag{II.13}$$

The form of this amplitude corresponds to the right-mover contribution of the closed string amplitude (9.1) before taking the γ_i to depend on the θ_i and integrating over the θ_i.

Appendix III. The KLT Formalism

In the text, we have presented the N-point massless-vector amplitude (2.1) using the Kawai, Lewellen and Tye (KLT) [12] formalism for internal Bardakci-Halpern [24] world-sheet fermions. These internal fermion contributions require no modification for use in the off-shell amplitudes considered in this paper. In the KLT formalism, the boundary conditions for the complex world-sheet fermions are represented by vectors $W_i = (l_1 \ldots l_{22} | r_1 \ldots r_{10})$, where the l_i component signifies that the ith left-mover fermion picks up an $\exp(-2\pi l_i)$ phase when going around the appropriate (world-sheet space or time) closed loop. A model specified by a set of basis vectors W_i is a consistent string theory if it satisfies certain constraint equations, eqs. (3.33–35) of ref. [12]. The space boundary-condition vectors specify the sectors, while the time boundary-condition vectors determine the generalized GSO projections that constrain the spectrum. The mass squared of a given state is determined by adding the quanta of the fermionic world-sheet oscillators to the vacuum energy with the usual left-right level-matching. Modular invariance requires that in calculating the partition function (or scattering amplitudes) we sum over time- and space-boundary conditions, with coefficients given in eq. (3.32) of ref. [12]. In the N-gluon amplitude (2.1) these coefficients are the $C^{\vec{\alpha}}_{\vec{\beta}}$.

Appendix IV. One-loop Closed String Conventions

We define theta functions for general twisted boundary conditions by

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu\,|\,\tau) = \sum_{n\in Z} e^{\pi i (n+\alpha-1/2)^2 \tau} e^{2\pi i (n+\alpha-1/2)(\nu-\beta-1/2)} \tag{IV.1}$$

while the Dedekind η function is

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau}\right) \tag{IV.2}$$

The bosonic partition function is

$$\mathcal{Z}_B(\tau) = \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-d} \tag{IV.3}$$

where d is the number of spacetime dimensions.

We define $\mathcal{Z}_1\left[{\alpha \atop \beta}\right](\tau)$ to be the partition function for a single left-moving complex fermion with $\left[{\alpha \atop \beta}\right]$ boundary conditions,

$$\mathcal{Z}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \operatorname{Tr}\left[e^{2\pi i \hat{H}_a\tau}e^{2\pi i(\frac{1}{2}-\beta)\hat{N}_a}\right] = \frac{e^{-2\pi i(1/2-\alpha)(1/2+\beta)}\vartheta\left[{\alpha \atop \beta}\right](0\,|\,\tau)}{\eta(\tau)} \tag{IV.4}$$

where the phase is present in order to be consistent with the KLT definition [12]. It is really irrelevant, and could be absorbed into the definitions of the summation coefficients $C^{\vec{\alpha}}_{\vec{\beta}}$.

Putting the pieces together, the complete partition function for the set of fermions with $\left[{\vec{\alpha} \atop \vec{\beta}}\right]$ boundary conditions is

$$\begin{aligned}\mathcal{Z}\begin{bmatrix}\vec{\alpha}\\ \vec{\beta}\end{bmatrix}(\tau) &= \mathcal{Z}_B(\tau)\prod_{i=1}^{\operatorname{len}\alpha_L}\mathcal{Z}_1\begin{bmatrix}\alpha_{Li}\\ \beta_{Li}\end{bmatrix}(\tau)\prod_{i=1}^{\operatorname{len}\alpha_R}\overline{\mathcal{Z}_1}\begin{bmatrix}\alpha_{Ri}\\ \beta_{Ri}\end{bmatrix}(\tau)\\ &= \left(\eta(\tau)\overline{\eta}(\tau)\sqrt{\operatorname{Im}\tau}\right)^{2-d}\\ &\quad\prod_{i=1}^{\operatorname{len}\alpha_L}\frac{e^{-2\pi i(1/2-\alpha_{Li})(1/2+\beta_{Li})}\vartheta\left[{\alpha_{Li} \atop \beta_{Li}}\right](0\,|\,\tau)}{\eta(\tau)}\prod_{i=1}^{\operatorname{len}\alpha_R}\frac{e^{2\pi i(1/2-\alpha_{Ri})(1/2+\beta_{Ri})}\overline{\vartheta}\left[{\alpha_{Ri} \atop \beta_{Ri}}\right](0\,|\,\tau)}{\overline{\eta}(\tau)}\end{aligned} \tag{IV.5}$$

The bosonic Green function is given by

$$G_B(\nu) = -\frac{1}{\pi}\ln\left|2\pi e^{-\pi(\operatorname{Im}\nu)^2/\operatorname{Im}\tau}\frac{\vartheta\left[{0 \atop 0}\right](\nu\,|\,\tau)}{\vartheta'\left[{0 \atop 0}\right](0\,|\,\tau)}\right| \tag{IV.6}$$

In a slight abuse of notation, we write the correlation function for right-movers as $G_B(\overline{\nu})$ although in fact it is equal to $\overline{G_B(\nu)}$.

A dotted variable, for our purposes, will always be taken to signify differentiation with respect to $\overline{\nu}$, so that

$$\dot{G}_B(\overline{\nu}) = \frac{i\operatorname{Im}\nu}{\operatorname{Im}\tau} - \frac{1}{2\pi}\frac{\vartheta'\left[{0 \atop 0}\right](\overline{\nu}\,|\,-\overline{\tau})}{\vartheta\left[{0 \atop 0}\right](\overline{\nu}\,|\,-\overline{\tau})} \tag{IV.7}$$

The fermionic particle correlation function $G_F\left[{\alpha \atop \beta}\right](\nu)$ and anti-particle correlation function $\hat{G}_F\left[{\alpha \atop \beta}\right](\nu)$ are defined as follows (excluding the case $\alpha = \beta = 0$):

$$\begin{aligned}G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= \frac{\vartheta\left[{\alpha \atop \beta}\right](\nu\,|\,\tau)\vartheta'\left[{0 \atop 0}\right](0\,|\,\tau)}{2\pi\vartheta\left[{0 \atop 0}\right](\nu\,|\,\tau)\vartheta\left[{\alpha \atop \beta}\right](0\,|\,\tau)}\\ \hat{G}_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &= G_F\begin{bmatrix}1-\alpha\\ 1-\beta\end{bmatrix}(\nu) = -G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(-\nu)\end{aligned} \tag{IV.8}$$

where the last equality derives from a theta function identity.

Under $\tau \to \tau + 1$,

$$\begin{aligned}
\mathcal{Z}_B(\tau) &\to \mathcal{Z}_B(\tau) \\
\mathcal{Z}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) &\to e^{i\pi(\alpha^2-\alpha+1/6)}\mathcal{Z}_1\begin{bmatrix}\alpha\\ \beta-\alpha\end{bmatrix} \\
G_B(\nu) &\to G_B(\nu) \\
\dot{G}_B(\overline{\nu}) &\to \dot{G}_B(\overline{\nu}) \\
G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &\to G_F\begin{bmatrix}\alpha\\ \beta-\alpha\end{bmatrix}(\nu)
\end{aligned} \tag{IV.9}$$

and under $\tau \to -1/\tau$,

$$\begin{aligned}
\mathcal{Z}_B(\tau) &\to \mathcal{Z}_B(\tau) \\
\mathcal{Z}_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) &\to e^{2\pi i(\alpha-1/2)(\beta-1/2)}\mathcal{Z}_1\begin{bmatrix}\beta\\ -\alpha\end{bmatrix} \\
G_B(\nu) &\to G_B(\nu) + \frac{\ln|\tau|}{\pi} \\
\dot{G}_B(\overline{\nu}) &\to (-\overline{\tau})\,\dot{G}_B(\overline{\nu}) \\
G_F\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\nu) &\to (-\tau)\,G_F\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}(\nu)
\end{aligned} \tag{IV.10}$$

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